Elements of Calculus | Session 3 (Week 2 A)

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Lecture notes from Spring 2025

2. Limits and Continuity

2.1 Sequences and Limits

Topic for Week 2 A: Sequences, Limits, Cauchy Sequences

Consider a sequence of numbers $(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots)$, e.g., $a_n = n$, or $a_n = \frac{1}{n^2}$.

A given sequence might become arbitrarily close to a certain number. If this is the case, we call this number the limit of the sequence. Limits are the central concept of Analysis (and Calculus). Let us make the concept of limit more precise.

Definition: We say that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to a if:

$$\underbrace{\forall \varepsilon > 0}_{\text{arbitrarily precision}} \quad \underbrace{\exists N \in \mathbb{N}}_{\substack{\text{large enough } N \\ \text{(can depend on } \varepsilon)}} \text{ s.t. } \underbrace{|a_n - a| < \varepsilon}_{\substack{a_n \text{ becomes arbitrarily} \\ \text{close to } a}} \quad \underbrace{\forall n \geq N.}_{\text{if } n \text{ is large enough}}$$

In this case we call a the **limit of $(a_n)_{n\in\mathbb{N}}$**, and write $a = \lim_{n\to\infty} a_n$ or $a_n \xrightarrow{n\to\infty} a$.

If no such a exists, we say that the sequence $(a_n)_{n\in\mathbb{N}}$ **diverges**.

Examples:

- $a_n = \frac{1}{n}$. For any $\varepsilon > 0$ we can choose $N = \lceil \frac{1}{\varepsilon} \rceil + 1$ (with $\lceil x \rceil$ the "ceiling fct." i.e., the least integer greater than x). $\Rightarrow |a_n - 0| < \varepsilon \ \forall n \geq N$, and hence $\lim_{n\to\infty} \frac{1}{n} = 0$.

- $a_n = (-1)^n$ is not convergent.

The following limit laws follow directly from the definition.

Proposition: If $a_n \xrightarrow{n\to\infty} a$ and $b_n \xrightarrow{n\to\infty} b$, then
- $a_n + b_n \xrightarrow{n\to\infty} a+b$
- $a_n b_n \xrightarrow{n\to\infty} ab$
- if $b_n \neq 0$ for large enough n and $b \neq 0$, $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \frac{a}{b}$.

Note that all convergent sequences are bounded, i.e., if $a_n \xrightarrow{n\to\infty} a$, then $\exists B$ s.t. $|a_n| \leq B \ \forall n \in \mathbb{N}$.

One more property:

Proposition (Squeeze or Sandwich Thm.): Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ with $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} c_n$. Then also $\lim_{n\to\infty} b_n = x$.

Example: $\lim_{n\to\infty} \frac{1}{n}\sin(n) = ?$ Cannot use limit laws since $\lim_{n\to\infty} \sin(n)$ does not exist.

But $-1 \leq \sin(n) \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{1}{n}\sin(n) \leq \frac{1}{n}$, and since $\pm\frac{1}{n} \xrightarrow{n\to\infty} 0$, we have $\frac{1}{n}\sin(n) \xrightarrow{n\to\infty} 0$.

In Mathematics, a fundamental role is played by the Cauchy sequences:

<u>Definition:</u> A sequence $(a_n)_{n\in\mathbb{N}}$ is called Cauchy sequence if:

$$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \text{ s.t. } |a_n-a_m|<\varepsilon \quad \forall n,m\geq N.$$

Note that any convergent sequence is a Cauchy sequence since $|a_n-a_m| = |a_n-a+(a-a_m)|$
$\leq |a_n-a|+|a_m-a|$.

But not every Cauchy sequence is convergent. The advantage of Cauchy sequences is that one can def. new things as their limit. This is indeed one way to construct $\mathbb{R}$. We consider all Cauchy sequences in $\mathbb{Q}$. Then any $x\in\mathbb{R}\setminus\mathbb{Q}$ can be constructed as the limit of a Cauchy sequence.

Ex.: let $x_0\in\mathbb{Q}$, and define iteratively $x_{n+1}=\frac{1}{2}\left(x_n+\frac{2}{x_n}\right)$.

Assuming the sequence $(x_n)_{n\geq0}$ converges (which it does), say $x_n \xrightarrow{n\to\infty} x^*$, then
$x^*=\frac{1}{2}\left(x^*+\frac{2}{x^*}\right) \Rightarrow \frac{1}{2}x^*=\frac{1}{x^*} \Rightarrow (x^*)^2=2$, i.e., $x^*=\sqrt{2}$.

Next: For given $x\in\mathbb{R}$, consider the sequence $a_n=\left(1+\frac{x}{n}\right)^n$.

We will show later that this sequence indeed has a limit: $a_n \xrightarrow{n\to\infty} e^x$, the exponential fct., with $e=2.718\ldots$ Euler's constant.

$\Rightarrow$ This defines e^x for any $x\in\mathbb{R}$. The inverse is called natural logarithm $\ln(x)$ ($\ln e^x = x$).

$\Rightarrow$ We define $a^x = e^{\ln a^x} = e^{x\ln a}$ for any $0<a\in\mathbb{R}$.

Next: Consider the sequence $(-1)^n$. It does not have a limit, but one would sometimes still like to talk about 1 as an "upper" limit and -1 as a "lower" limit. This is done in the following way.

Definition: Let $A \subset \mathbb{R}$. We def.:

- the supremum of A as the smallest upper bound for A: $\sup A := \{\text{smallest } x \in \mathbb{R} : a \leq x \ \forall a \in A\}$,
- the infimum of A as the biggest lower bound for A: $\inf A$: $\{\text{biggest } x \in \mathbb{R} : x \leq a \ \forall a \in A\}$.

Ex.: • $\sup\{x \in \mathbb{R} : 0 < x < 2\} = 2 = \sup(0,2)$ (but note $2 \notin (0,2)$).

• $\inf(0,2) = 0$

• $\sup\{\frac{1}{n} : n \in \mathbb{N}\} = 1$, $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$.

For limits, we then def.:

Definition: Let $(a_n)_{n \in \mathbb{N}}$ be a real sequence.

- The limit superior is def. as $\limsup\limits_{n \to \infty} a_n = \lim\limits_{m \to \infty} \sup\limits_{n \geq m} a_n$.
- The limit inferior is def. as $\liminf\limits_{n \to \infty} a_n = \lim\limits_{m \to \infty} \inf\limits_{n \geq m} a_n$.

Ex.: • $\limsup\limits_{n \to \infty} (-1)^n = 1$, $\liminf\limits_{n \to \infty} (-1)^n = -1$

Note: limsup and liminf either exist or are $\pm\infty$. If $\limsup a_n = a = \liminf a_n$, then $\lim\limits_{n \to \infty} a_n = a$.