Elements of Calculus

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Lecture notes from Spring 2025

Session 4
(Week 2B)

2. Limits and Continuity

2.2 Series and Power Series

Topic for Week 2B: Series and Convergence Tests, Power Series and Radius of Convergence

First, we consider infinite series.

Let us consider $a_0 + a_1 + a_2 + \dots + a_N = \sum_{k=0}^{N} a_k =: S_N$, called **partial sum**.

Ex.: • Arithmetic series: $\sum_{k=0}^{N} k$

We find $\sum_{k=0}^{N} k = 0 + 1 + 2 + \dots + (N-1) + N = (N+1)\frac{N}{2}$

(The story is that Gauß figured this out in elementary school.)

• Geometric series: $\sum_{k=0}^{N} x^k = ?$

We compute: $\sum_{k=0}^{N} x^k - x\sum_{k=0}^{N} x^k = \underbrace{\sum_{k=0}^{N} x^k}_{=1+x+x^2+\dots+x^N} - \underbrace{\sum_{k=0}^{N} x^{k+1}}_{=x+x^2+\dots+x^N+x^{N+1}} = 1 - x^{N+1}$

$$\sum_{k=0}^{N} x^k - x\sum_{k=0}^{N} x^k = (1-x)\sum_{k=0}^{N} x^k$$

$$\Rightarrow \boxed{\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}}$$

Back to $S_N = \sum_{k=0}^{N} a_k$.

Observation: $(S_N)_{N\in\mathbb{N}}$ is a sequence

$\Rightarrow$ it is either $\rightarrow$ convergent, i.e., $\lim_{N\to\infty} S_N =: \sum_{k=0}^{\infty} a_k$ exists

$\searrow$ or divergent

Ex.: $\sum_{k=0}^{\infty} x^k = \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} = \frac{1-\lim_{N\to\infty} x^{N+1}}{1-x} = \begin{cases} \text{convergent to } \frac{1}{1-x} \text{ for } -1<x<1 \\ \text{divergent to } +\infty \text{ for } x\geq 1 \\ \text{divergent for } x\leq -1 \end{cases}$

There are several criteria to determine whether $\sum_{k=0}^{\infty} a_k$ is convergent or not:

- Necessary condition: $\lim_{k\to\infty} a_k = 0$

  Ex.: $\sum_{k=0}^{\infty} k^{\frac{3}{2}}$ or $\sum_{k=0}^{\infty} \frac{k}{k+1}$ are surely divergent.

- $\boxed{\text{Comparison test}}$: Let $0\leq a_k \leq b_k \quad \forall k\in\mathbb{N}$

  $\hookrightarrow$ If $\sum_{k=0}^{\infty} b_k$ converges, then so does $\sum_{k=0}^{\infty} a_k$

  $\hookrightarrow$ If $\sum_{k=0}^{\infty} a_k$ diverges, then so does $\sum_{k=0}^{\infty} b_k$

  Ex.: $b_k = \frac{1}{k+1}$, i.e., $\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \frac{1}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$

  $\hookrightarrow$ compare with $\sum_{k=0}^{\infty} a_k = 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=\frac{1}{2}} + \dots \Rightarrow$ diverges

  Thus also $\sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges.

• $\boxed{\text{Ratio test}}$: If $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ is $\begin{cases} <1, & \text{then series converges} \\ >1 \text{ or } \infty, & \text{then series diverges} \\ =1 \text{ or doesn't exist}, & \text{then test is inconclusive} \end{cases}$

Ex.: • $\sum\limits_{k=0}^{\infty}\frac{x^k}{k!}$ for $x\in\mathbb{R}$

$$\lim_{k\to\infty}\left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\ \frac{k!}{(k+1)!}\right| = \lim_{k\to\infty}\frac{|x|}{k+1} = 0 \quad \forall x\in\mathbb{R}$$

$\Rightarrow$ Series converges

• $\sum\limits_{k=0}^{\infty}\frac{1}{k+1}$ $\to$ we find $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\frac{1/k+2}{1/k+1} = \lim\limits_{k\to\infty}\frac{k+1}{k+2} = 1$

$\Rightarrow$ test inconclusive

Next: Power Series

Definition: $\sum\limits_{k=0}^{\infty} a_k x^k$ is called $\boxed{\text{power series.}}$

$\rho := \sup\left\{|x| : \sum\limits_{k=0}^{\infty} a_k x^k \text{ converges}\right\}$ is called $\boxed{\text{radius of convergence.}}$

So by definition, $\sum\limits_{k=0}^{\infty} a_k x^k$ converges if $-\rho < x < \rho$.

Ratio test: convergence if $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| |x| < 1$

$\Rightarrow$ need $|x| < \lim\limits_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$ $\Rightarrow$ $\boxed{\rho = \lim\limits_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| \text{ if this limit exists}}$

Ex.: • $\sum_{k=0}^{\infty} (2x)^k = \sum_{k=0}^{\infty} \underbrace{2^k}_{=a_k} x^k$

we find $\rho = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \frac{2^k}{2^{k+1}} = \frac{1}{2}$

$\Rightarrow$ series converges for $-\frac{1}{2} < x < \frac{1}{2}$ (but note: does not converge <u>at</u> $x = \frac{1}{2}$)

• above we found that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges $\forall x \in \mathbb{R}$, i.e., $\rho = \infty$

Note: ρ can be 0, some number >0, or ∞