Elements of Calculus

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Lecture notes from Spring 2025

Example Session Week 2

Example Session for:

Week 2 A: Sequences, Limits, Cauchy Sequences

Week 2 B: Series and Convergence Tests, Power Series and Radius of Convergence

Examples of limits

- $\lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{n+1-1}{n+1} = \lim\limits_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \overset{\text{limit law}}{=} 1$

- $\lim\limits_{n\to\infty} \frac{2n^2-1}{n^2+1} = \lim\limits_{n\to\infty} \frac{2-\frac{1}{n^2}}{1+\frac{1}{n^2}} \overset{\text{limit law}}{=} 2$

- For $q \in \mathbb{R}$, $\lim\limits_{n\to\infty} q^n = \begin{cases} 0 & \text{for } |q|<1 \\ 1 & \text{for } q=1 \\ \text{diverges to } \infty & \text{for } q>1 \\ \text{diverges} & \text{for } q \le -1 \end{cases}$

- $\lim\limits_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$

- $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n+\frac{1}{3}} = \lim\limits_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} \sqrt{n+\frac{1}{3}}$

$$= \lim_{n\to\infty} (n+1-n) \frac{\sqrt{n+\frac{1}{3}}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{3n}}}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

The Exponential Function

We consider the sequence $\left(1+\frac{x}{n}\right)^n$.

The binomial theorem yields:

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{n}\right)^k 1^{n-k}$$

$$= \sum_{k=0}^{n} \underbrace{\frac{n!}{k!\,(n-k)!}\,\frac{1}{n^k}} x^k$$

$$= \frac{1}{k!}\,\frac{n(n-1)\cdots(n-k+1)}{n\ n\ \cdots\ n} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(k-1)}{n}\right)$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(k-1)}{n}\right)$$

One can show that this implies indeed $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ which exists by the ratio test $\forall x \in \mathbb{R}$.

$\Longrightarrow$ We def. $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!}$.

Note that this def. satisfies the exponential law as required:

$$e^x e^y = \sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{m=0}^{\infty}\frac{y^m}{m!} = \left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots\right)\left(1+y+\frac{y^2}{2}+\frac{y^3}{3!}+\dots\right)$$

$$= \left(1+x+y+\frac{x^2}{2}+\frac{y^2}{2}+xy+\dots\right)$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\frac{x^{n-\ell}}{(n-\ell)!}\,\frac{y^\ell}{\ell!}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}y^{\ell}}_{(x+y)^n} = e^{x+y}$$