Elements of Calculus

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Lecture notes from Spring 2025

Session 5 (Week 3 A)

2. Limits and Continuity

2.3 Limits of Functions

Topic for Week 3A: Limits of Functions and Asymptotes

Next: Limits of functions.

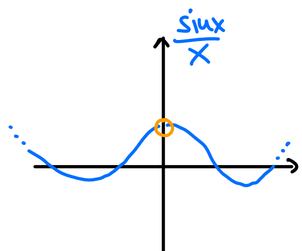
Definition: Let $x_0 \in (a,b)$ and $f:(a,b)\setminus\{x_0\} \to \mathbb{R}$. Then $\lim_{x\to x_0} f(x) = L$ (or $f(x) \xrightarrow[x\to x_0]{} L$) if for all sequences $(x_n)_{n\in\mathbb{N}}$ in $(a,b)\setminus\{x_0\}$ with limit x_0 we have $\lim_{n\to\infty} f(x_n) = L$.

In other words: $\lim_{x\to x_0} f(x) = L$ if

$$\underbrace{\forall \varepsilon>0}_{\text{precision for } f(x)}\ \underbrace{\exists \delta>0}_{\text{precision for } x}\ \text{s.t.}\ \underbrace{\forall x\in(a,b)\setminus\{x_0\} \text{ with } |x-x_0|<\delta}_{x \text{ is } \delta\text{-close to } x_0}\ \text{we have}\ \underbrace{|f(x)-L|<\varepsilon}_{f(x) \text{ becomes arbitrarily close to } L}.$$

Examples:

- $f:\mathbb{R}\setminus\{0\}\to\mathbb{R},\ x\mapsto \frac{\sin x}{x}$. A plot reveals:



It looks like $\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1$, we will show this more precisely later.

- $f:\mathbb{R}\setminus\{0\}\to\mathbb{R},\ x\mapsto \frac{1}{x}$. Here, $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Next: The limit laws for sequences also hold for functions:

Proposition:

(i) $\lim\limits_{x\to x_0} \big(f(x)+g(x)\big) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$,

(ii) $\lim\limits_{x\to x_0} f(x)g(x) = \lim\limits_{x\to x_0} f(x) \lim\limits_{x\to x_0} g(x)$,

(iii) $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$ if $\lim\limits_{x\to x_0} g(x) \neq 0$.

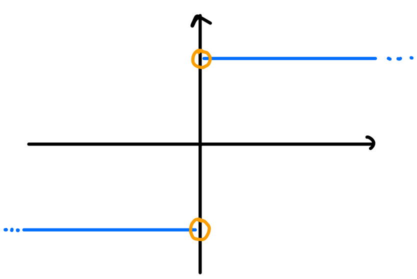
These laws can be easily proved from the definition above (or the limit laws for sequences).

Example: $$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \underset{\uparrow}{=} \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x+2}{x+3} \underset{\uparrow\,(iii)}{=} \frac{\lim_{x\to 2}(x+2)}{\lim_{x\to 2}(x+3)} = \frac{4}{5}.$$

Can't apply (iii) directly bc. denominator becomes zero

Next: Left and right limits.

Consider the following example: $f(x) = \dfrac{|x|}{x}$ with domain $\mathbb{R}\setminus\{0\}$:



Here, $\lim\limits_{x\to 0} f(x)$ does not exist, because for every $x>0$ we have

$|f(x)-f(-x)| = |1-(-1)| = 2$, i.e., it is not possible to find L s.t. $|f(x)-L|<\varepsilon$ $\forall \varepsilon>0$, no matter how close x is chosen to zero.

But it makes sense to define limits from the left or right:

$\lim\limits_{\substack{x\to 0\\ \boxed{x>0}}} f(x) = 1$ and $\lim\limits_{\substack{x\to 0\\ \boxed{x<0}}} f(x) = -1$ here.

↑ use only $x>0$ in the ε-δ-definition above

Definition:

(i) The limit from the right is $\lim\limits_{\substack{x\to x_0\\ x>x_0}} f(x) \equiv \lim\limits_{\underbrace{x\searrow x_0}} f(x) \equiv \lim\limits_{x\to x_0^+} f(x)$.

"limit from above"

(ii) The limit from the left is $\lim\limits_{\substack{x\to x_0\\ x<x_0}} f(x) \equiv \lim\limits_{\underbrace{x\nearrow x_0}} f(x) \equiv \lim\limits_{x\to x_0^-} f(x)$.

"limit from below"

Note: $\lim$ exists if and only if both left- and right-sided limits exist and coincide.

Next: Quantify limits as x or $f(x)\longrightarrow\infty$ better.

Asymptotes can capture the behavior of functions for very large x, e.g., $\frac{1}{x}$ tends to zero for larger and larger x.

Definition:

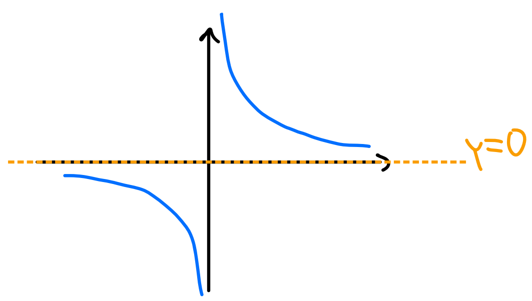
We write
- $\lim\limits_{\underbrace{x\to\infty}} f(x) = L$ if $\lim\limits_{y\searrow 0} f\left(\frac{1}{y}\right) = L$,

  "x to infinity"
- $\lim\limits_{x\to-\infty} f(x) = L$ if $\lim\limits_{y\nearrow 0} f\left(\frac{1}{y}\right) = L$,

and call the line $y=L$ a **horizontal asymptote**.

Examples:

- $\lim\limits_{x\to\infty} \frac{1}{x} = \lim\limits_{y\searrow 0} \frac{1}{1/y} = \lim\limits_{y\searrow 0} y = 0$, i.e., $y=0$ is the horizontal asymptote for $\frac{1}{x}$ as $x\to\infty$.



(Also $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$.)

- $$\lim_{x\to\infty} \frac{x^2+3x-2}{3x^2-2x+5} = \lim_{x\to\infty} \frac{1+\frac{3}{x}-\frac{2}{x^2}}{3-\frac{2}{x}+\frac{5}{x^2}} \underset{\substack{\uparrow \\ \text{by definition}}}{:=} \lim_{y\searrow 0} \frac{1+3y-2y^2}{3-2y+5y^2} \underset{\substack{\uparrow \\ \text{limit laws}}}{=} \frac{1}{3}$$

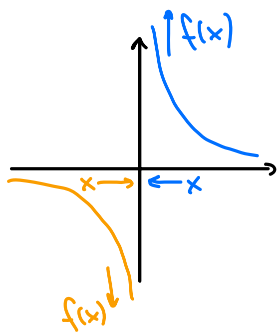
Definition:

If $f(x)$ increases (decreases) without bounds as $x \nearrow x_0$ $(x \searrow x_0)$, we write $\underbrace{\lim_{\substack{x\nearrow x_0 \\ (x\searrow x_0)}} f(x) = \infty \ (-\infty)}$ and say that f has a **vertical asymptote** $x = x_0$.

Still, we say that here the limit does not exist. We cannot compute, i.e., do algebraic manipulations with ∞.

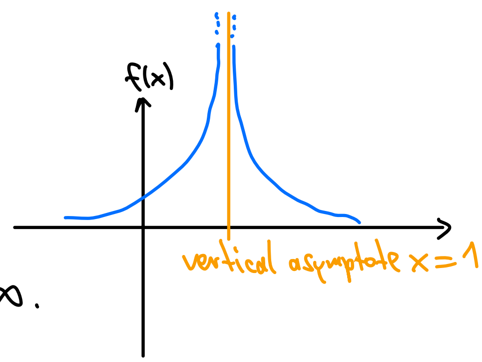
Examples:

- $\lim_{x\searrow 0} \frac{1}{x} = \infty$, $\lim_{x\nearrow 0} \frac{1}{x} = -\infty$.



- $\lim_{x\searrow 1} \frac{1}{(x-1)^2} = \infty$, $\lim_{x\nearrow 1} \frac{1}{(x-1)^2} = \infty$

$\Rightarrow$ Here we would thus write $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.



Finally: Two very important limits (proof in Example Session)

$$\boxed{\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0 \text{ for every } \alpha > 0, \text{ i.e., } e^x \text{ grows faster than any power of } x,}$$

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0 \text{ for every } \alpha > 0, \text{ i.e., } \ln x \text{ grows slower than any power of } x.}$$