Elements of Calculus

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Lecture notes from Spring 2025

Session 6
(Week 3B)

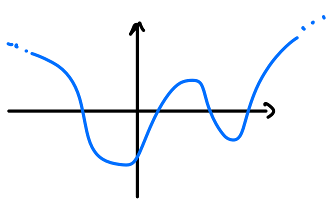
2. Limits and Continuity

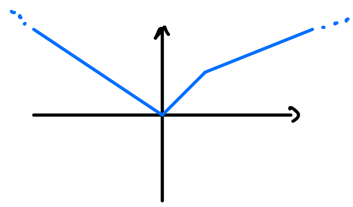
2.4 Continuity

Topic for Week 3B: Continuity and the Intermediate Value Theorem

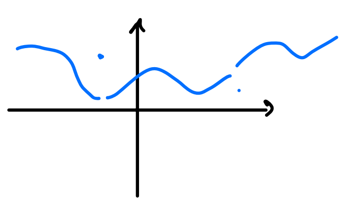
Informally: "A function is continuous if it can be drawn without lifting the pen".

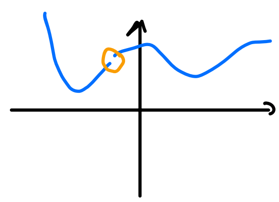
E.g., • continuous functions:



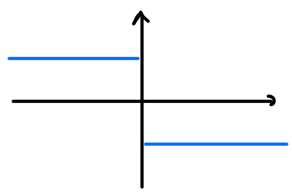


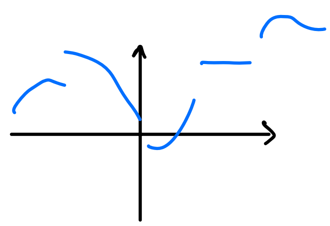
• functions with removable discontinuities:





• functions with non-removable discontinuities:





Formal definition with limits:

Definition:

A function $f:(a,b) \to \mathbb{R}$ is continuous at $x_0 \in (a,b)$ if $\lim\limits_{x \to x_0} f(x) = f(\overbrace{x_0}^{\lim_{x\to x_0} x})$.

If f is continuous at every point in its domain, we say "f is continuous".

Discontinuous functions can have:

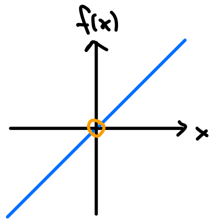
a) Removable discontinuities: $f(x_0) \neq L$ or $f(x_0)$ not defined, but $\lim_{x \to x_0} f(x) = L$.

Then we can remove the discontinuity by defining $f(x_0) := L$.

b) Non-removable discontinuities such as jumps or divergences.

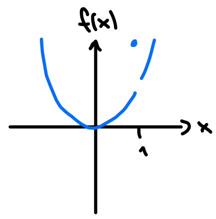
Simple examples:

- $f(x) = \frac{x^2}{x}$ with domain $\mathbb{R} \setminus \{0\}$.



We can remove the discontinuity by defining $f(0) := \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$.

- $f(x) = \begin{cases} x^2 & \text{for } x \neq 1, \\ 2 & \text{for } x = 1. \end{cases}$



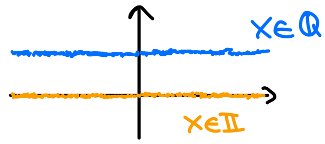
Discontinuity at $x = 1$ can be removed by setting $f(1) := \lim_{x \to 1} x^2 = 1$.

Note: Without proof, we can use that the following standard functions are continuous:

- polynomials $\sum_{k=0}^{n} a_k x^k$,
- trigonometric functions $\sin x$, $\cos x$, and $\tan x$ with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$,
- $\exp x = e^x$, $\ln x$ with domain $(0, \infty)$
- hyperbola $\frac{1}{x}$ with domain $\mathbb{R} \setminus \{0\}$. ($\frac{1}{x}$ is continuous at all $x \in \mathbb{R} \setminus \{0\}$, but it is discontinuous at $x = 0$, where it is not well-defined (vertical asymptote).)

Also note: There are functions on $\mathbb{R}$ that are discontinuous everywhere, e.g.,

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} = \mathbb{I}. \end{cases}$$



Once we know that a function is continuous, we can use lots of useful theorems.
(We state them here without proof, or with "proof by picture".)

__Theorem:__

If $f: \mathbb{R} \to \mathbb{R}$ is continuous, and $g: (a,b) \to \mathbb{R}$ has $\lim\limits_{x \to x_0} g(x) = L$ for $x_0 \in (a,b)$,
then $\lim\limits_{x \to x_0} f(g(x)) = f(L) = f\left(\lim\limits_{x \to x_0} g(x)\right)$.

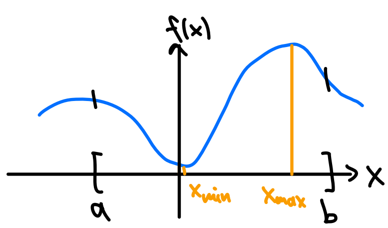
↳ i.e., we can move limits inside of continuous functions.

E.g., $\lim\limits_{x \to 3} \sin\left(\frac{x^2 - 9}{x-3}\right) = \sin\left(\lim\limits_{x\to 3} \underbrace{\frac{x^2-9}{x-3}}_{= \frac{(x-3)(x+3)}{x-3} = x+3}\right) = \sin(6)$.

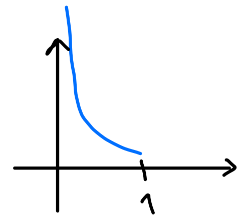
__Extreme Value Theorem:__

If $f: \underbrace{[a,b]}_{\text{closed interval is important here}} \to \mathbb{R}$ is continuous, then f assumes its minimum and maximum, i.e., there exist

- $x_{min} \in [a,b]$ s.t. $f(x) \geq f(x_{min})$ $\forall x \in [a,b]$,
- $x_{max} \in [a,b]$ s.t. $f(x) \leq f(x_{max})$ $\forall x \in [a,b]$.

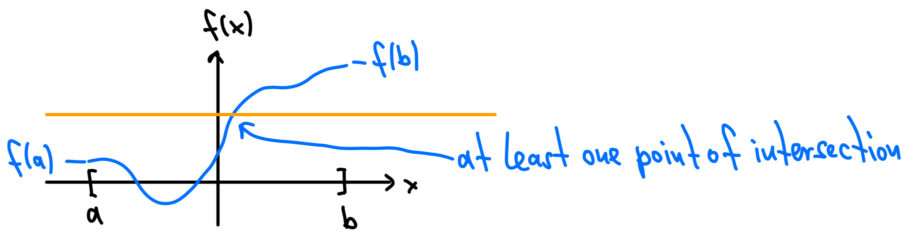


Ex.: For $f(x)=\frac{1}{x}$ with domain $(0,1)$, the theorem does not apply because the interval $(0,1)$ is not closed. Indeed, there is no maximum.



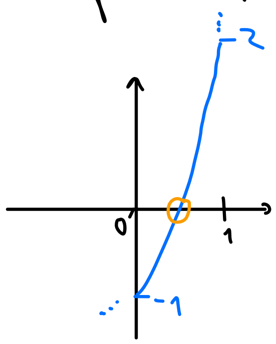
Intermediate Value Theorem:

If $f:[a,b]\to\mathbb{R}$ is continuous, then it assumes all intermediate values between $f(a)$ and $f(b)$, i.e., for every $y\in[f(a),f(b)]$ (or $[f(b),f(a)]$ if $f(b)<f(a)$) there is $x\in[a,b]$ s.t. $y=f(x)$.



Example: $p(x)=x^3+x^2+x-1$

We have $p(1)=1+1+1-1=2$, $p(0)=-1$, so we know p has at least one root in $[0,1]$.



This leads to the bisection method, see example session.

From the limit laws, we immediately get:

Proposition:

If f and g are continuous at x_0, then also $f+g$, $f\cdot g$, and $\frac{f}{g}$ for $g(x_0)\neq 0$, are continuous at x_0.

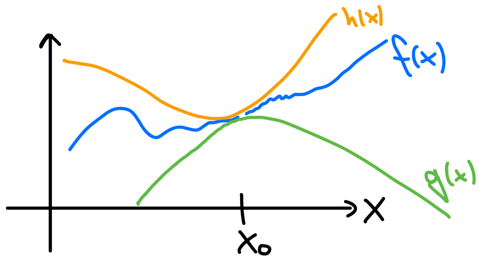
Also, $f(g(x)) =: (f\circ g)(x)$ is continuous at x_0 if g is cont. at x_0 and f is cont. at $g(x_0)$.

Furthermore, the squeeze law immediately implies:

Theorem:

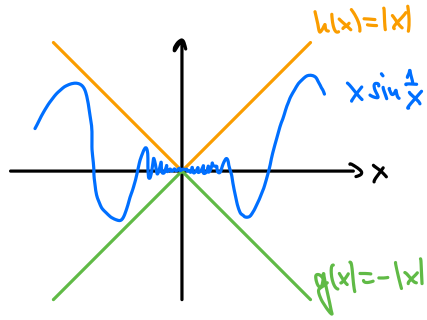
Let $x_0 \in (a,b)$, and $f, g, h : (a,b) \setminus \{x_0\} \to \mathbb{R}$ be such that $g(x) \leq f(x) \leq h(x)$ and

$\lim_{x \to x_0} g(x) \leq L \leq \lim_{x \to x_0} h(x)$. Then $\lim_{x \to x_0} f(x) = L$.



Example: $f(x) = x \sin \frac{1}{x}$ with domain $\mathbb{R} \setminus \{0\}$:

Note: $|f(x)| = |x| \underbrace{\left|\sin \frac{1}{x}\right|}_{\leq 1} \leq |x|$



$\Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$ and f is continuous on $\mathbb{R}$ with the definition $f(0) := 0$.