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Lecture notes from Spring 2025

Example Session for:

Week 3A: Limits of Functions and Asymptotes

Week 3B: Continuity and the Intermediate Value Theorem

Limits Involving the Exponential Function and the Logarithm

Question: For $\alpha > 0$, what is $\lim_{x\to\infty} x^{\alpha} e^{-x} = \lim_{x\to\infty} \frac{x^{\alpha}}{e^{x}}$? ← Both x^{α} and e^{x} tend to ∞.

Recall $e^{x} = \sum_{x=0}^{\infty} \frac{x^k}{k!}$

$$\Rightarrow \lim_{x\to\infty} \frac{x^{\alpha}}{\sum_{k=0}^{\infty} \frac{x^k}{k!}} = \lim_{x\to\infty} \frac{1}{x^{-\alpha} + x^{1-\alpha} + \frac{1}{2} x^{2-\alpha} + \dots} = 0$$

We conclude:

$$\boxed{\lim_{x\to\infty} \frac{x^{\alpha}}{e^{x}} = 0 \quad \text{for every } \alpha > 0 \text{, i.e., } e^{x} \text{ grows faster than any power of } x,}$$

Instead of $x \to \infty$ we can also set $x = \ln y$ and let $y \to \infty$. Then we find

$$0 = \lim_{y\to\infty} \frac{(\ln y)^{\alpha}}{e^{\ln y}} = \lim_{y\to\infty} \frac{(\ln y)^{\alpha}}{y} = \lim_{y\to\infty} \left(\frac{\ln y}{y^{\alpha}}\right)^{\frac{1}{\alpha}}$$

But if $\lim_{y\to\infty} f(y)^{\frac{1}{\alpha}} = 0$ then also $\lim_{y\to\infty} f(y) = 0$, since $x^{\frac{1}{\alpha}}$ is continuous!

We conclude:

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0 \text{ for every } \alpha > 0\text{, i.e., } \ln x \text{ grows slower than any power of } x.}$$

Note: Writing $x = \frac{1}{y}$, we find $0 = \lim_{y\to 0} \frac{\ln \frac{1}{y}}{\left(\frac{1}{y}\right)^\alpha} = \lim_{y\to 0} \frac{\overbrace{\ln 1}^{=0} - \ln y}{y^{-\alpha}} = -\lim_{y\to 0} y^\alpha \ln y$,

i.e., $\lim_{x\to 0} x^\alpha \ln x = 0$.

Limit Laws:

$$\lim_{t\to 0} \frac{\sqrt{t+25}-5}{t} = \lim_{t\to 0} \frac{(\sqrt{t+25}-5)(\sqrt{t+25}+5)}{t\,(\sqrt{t+25}+5)} = \lim_{t\to 0} \frac{t+25-25}{t(\sqrt{t+25}+5)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t+25}+5} = \frac{1}{\lim_{t\to 0}\sqrt{t+25}+5} = \frac{1}{5+5} = \frac{1}{10}.$$

use limit laws (i) and (iii)

$\lim_{t\to 0}\sqrt{t+25} = \sqrt{25}$ follows from continuity of $\sqrt{x}$

Asymptotes:

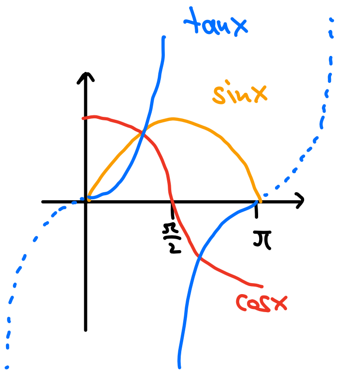
- $\lim_{x\to\infty} \frac{x^2+3x-2}{3x^2-2x+5} = \lim_{x\to\infty} \frac{1+\frac{3}{x}-\frac{2}{x^2}}{3-\frac{2}{x}+\frac{5}{x^2}} := \lim_{y\searrow 0} \frac{1+3y-2y^2}{3-2y+5y^2} = \frac{1}{3}$

by definition

Limit law (iii)

- $\lim_{x \nearrow \frac{\pi}{2}} \tan x = \lim_{x \nearrow \frac{\pi}{2}} \frac{\sin x}{\cos x} = \infty$

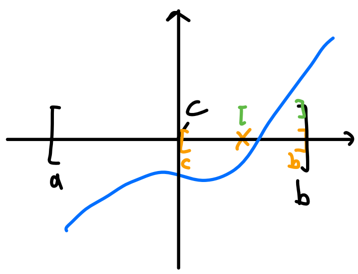
  $\lim_{x \searrow \frac{\pi}{2}} \tan x = -\infty$



Bisection Method:

Suppose $f: [a,b] \to \mathbb{R}$ is continuous, and $f(a) < 0$, $f(b) > 0$ (or the other way around). Then by the intermediate value theorem, f has a root in $[a,b]$.

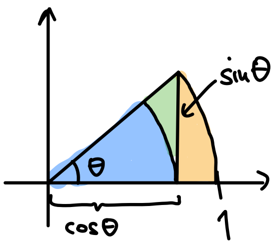
Then:
- Check $f(c)$, with c the midpoint of the interval, i.e., $c = \frac{a+b}{2}$.
- If $f(c) < 0$, $\exists$ root in $[c,b]$; if $f(c) > 0$, $\exists$ root in $[a,c]$.
- Repeat until necessary precision is reached.



Application of Squeeze Law:

Let $f(\theta) = \dfrac{\sin\theta}{\theta}$. What is $\lim\limits_{\theta \to 0} f(\theta)$?

Consider the following picture:



The areas are: $A_{b+g} = \frac{1}{2} \cos\theta \sin\theta$ (blue + green)

$$A_b = \underbrace{\frac{\theta}{2\pi}}_{\text{fraction of the circle, "cake piece"}} \underbrace{\pi r^2}_{\text{area of circle with radius } r} = \frac{1}{2}\theta \cos^2\theta \quad \text{(blue)}$$

$$A_{b+g+o} = \frac{1}{2}\theta \quad \text{(blue + green + orange)}$$

We have $A_b \leq A_{b+g} \leq A_{b+g+o}$, i.e., $\frac{1}{2}\theta\cos^2\theta \leq \frac{1}{2}\cos\theta\sin\theta \leq \frac{1}{2}\theta$

$$\Rightarrow \underbrace{\cos\theta}_{\to 1 \text{ as } \theta \to 0} \leq \frac{\sin\theta}{\theta} \leq \underbrace{\frac{1}{\cos\theta}}_{\to 1 \text{ as } \theta \to 0}$$

$$\Rightarrow \boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.}$$