Elements of Calculus

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Lecture notes from Spring 2025

Session 7
(Week 4 A)

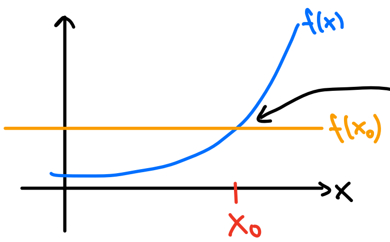
3. Differentiation in One Variable

3.1 Definition, Properties, and Examples

Topic for Week 4 A: Definition of Differentiation and Differentiation Rules

Task: We would like to approximate a function $f:(a,b)\to\mathbb{R}$ near a point $x_0\in(a,b)$.

First idea: Approximation by a constant: $f(x)\approx f(x_0)$ if x near x_0.

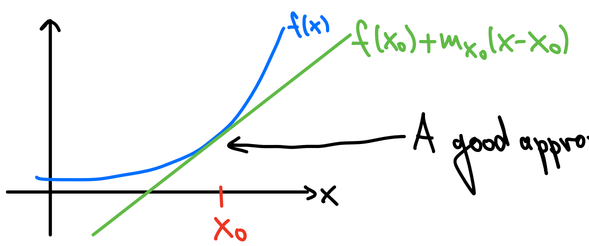


Quite large error when $x \neq x_0$.

This is a linear fct. in $x-x_0$ from $\mathbb{R}\to\mathbb{R}$.

A better idea: Add to the approximation a linear function: $f(x)\approx \underbrace{f(x_0)+\overbrace{m_{x_0}(x-x_0)}}$, with some $m_{x_0}\in\mathbb{R}$.

As a fct. of x, we would call this an affine linear fct. (i.e., a linear fct. plus a constant).



A good approximation when $x\approx x_0$

If such an affine linear approximation is possible at x_0, we call the function differentiable at x_0, and we call the corresponding m_{x_0} the derivative of f at x_0.

This is summarized in the following definition:

Definition: A fct. $f:(a,b)\to\mathbb{R}$ is called differentiable at $x_0\in(a,b)$ if there exists $m_{x_0}\in\mathbb{R}$ s.t.

$$f(x)=f(x_0)+m_{x_0}(x-x_0)+r_{x_0}(x) \quad \text{and } r_{x_0}(x) \text{ satisfies } \underbrace{\left|\frac{r_{x_0}(x)}{x-x_0}\right| \xrightarrow[x\to x_0]{} 0}_{\text{The rest term goes to 0 faster than linear.}}.$$

In this case we call f differentiable at x_0 and we call m_{x_0} the derivative of f at x_0.
We usually write $\boxed{m_{x_0}=f'(x_0) \text{ or } m_{x_0}=\frac{df}{dx}(x_0)}$.

If f is differentiable for all $x_0\in(a,b)$, we call f differentiable.

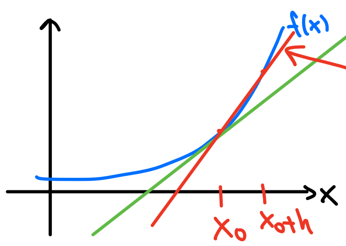
Remarks:

- We can rewrite this def. as: $$m_{x_0}=\frac{f(x)-f(x_0)}{x-x_0}-\underbrace{\frac{r_{x_0}(x)}{x-x_0}}_{\xrightarrow[x\to x_0]{} 0 \text{ if } f \text{ differentiable at } x_0}$$

  Hence $f'(x_0)=\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$. This is the slope of f at x_0.

  More precisely: If this limit exists, f is differentiable at x_0 and this limit is the derivative.

  Note that we can write $x-x_0=h$, and then $\boxed{f'(x_0)=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}}$.



line $y=f(x_0)+\frac{f(x_0+h)-f(x_0)}{h}(x-x_0)$
approaches green line as $h\to 0$.

- Suppose f is differentiable at x_0. Then $\lim\limits_{x\to x_0}f(x)=\lim\limits_{x\to x_0}\left(f(x_0)+m_{x_0}(x-x_0)+r_{x_0}(x)\right)=f(x_0)$, i.e., f is continuous at x_0.
  (Differentiability implies continuity, but not the other way around.)

Examples:

- $f(x)=x^2$. We compute:

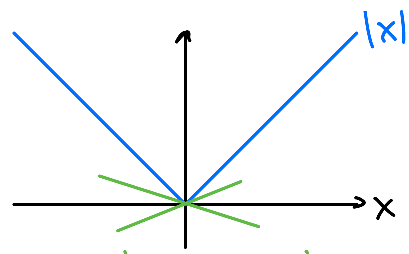
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}(2x+h)=2x.$$

$\Rightarrow f(x)=x^2$ is differentiable at any $x\in\mathbb{R}$ and $f'(x)=2x$.

- $f(x)=|x|$ at $x_0=0$. We compute:

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{|h|}{h} \text{ does not exist.}$$

$\Rightarrow f(x)=|x|$ is not differentiable at 0.



No tangent line approximation possible at 0.

Many more examples are discussed in the Example Session.

Next, we discuss some standard differentiation rules:

- If f' and g' exist, then $\boxed{(f+g)'=f'+g'}$ (Follows directly from the limit laws.)

- If f' and g' exist, then $\boxed{(fg)'=f'g+fg' \quad \text{(Product Rule.)}}$

Proof: $$(fg)'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

limit of sum = sum of limits

$$=\lim_{h\to 0}f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\frac{f(x+h)-f(x)}{h}$$

limit of product = product of limits and $\lim_{h\to 0}f(x+h)=f(x)$ since f is continuous

$$=f(x)g'(x)+g(x)f'(x)$$

- If f' and g' exist and $g \neq 0$, then $\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \text{(Quotient Rule)}}$

Proof: $f = \frac{f}{g} g \Rightarrow f' = \left(\frac{f}{g} g\right)' \underset{\text{product rule}}{=} \left(\frac{f}{g}\right)' g + \frac{f}{g} g'$

$$\Rightarrow \left(\frac{f}{g}\right)' = \frac{f' - \frac{f}{g} g'}{g} = \frac{f'g - fg'}{g^2}$$

- If f' and g' exist, then $\boxed{\left(f(g(x))\right)' = \underbrace{f'(g(x))}_{\text{"outer derivative"}} \underbrace{g'(x)}_{\text{"inner derivative"}} \quad \text{(Chain Rule)}}$

Proof: $$\left(f(g(x))\right)' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{= \frac{f(g(x)+k(h)) - f(g(x))}{k(h)}} \underbrace{\frac{g(x+h) - g(x)}{h}}_{\xrightarrow[h\to 0]{} g'(x)}$$

with $k(h) := g(x+h) - g(x)$

Note: $\lim_{h\to 0} k(h) = \lim_{h \to 0} (g(x+h) - g(x)) = 0$ since g continuous.

$$\xrightarrow[h\to 0]{} f'(g(x))$$

- Let $f: (a,b) \to \mathbb{R}$ be differentiable and invertible on its range with inverse f^{-1}. Let $f(x_0) = y_0$.

Then $\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} \quad \text{(Inverse Function Rule).}}$

Proof: $f^{-1}(f(x)) = x \Rightarrow (f^{-1})'(f(x))\, f'(x) = 1 \Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}$

• Let $f(x)=\sum_{k=0}^{\infty} a_k x^k$ have radius of convergence ϱ.

Then $f'(x)=\sum_{k=1}^{\infty} a_k (x^k)' = \sum_{k=1}^{\infty} a_k k x^{k-1}$, and this power series has the same radius of convergence ϱ.

$$= \lim_{h\to 0} \frac{(x+h)^k - x^k}{h} = \lim_{h\to 0} \frac{\sum_{m=0}^{k} \binom{k}{m} x^m h^{k-m} - x^k}{h} = \binom{k}{1} x^{k-1} + 0 = kx^{k-1}$$

Why? Assuming the ratio test works, we have $\varrho_{f'} = \lim_{k\to\infty} \left| \frac{k a_k}{(k+1) a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \varrho_f$.