Elements of Calculus

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Lecture notes from Spring 2025

Session 8
(Week 4B)

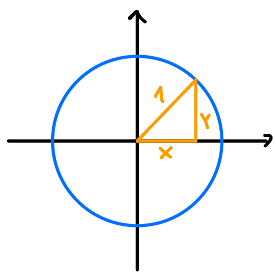
3. Differentiation in One Variable

3.1 Definition, Properties, and Examples

Topic for Week 4B: Implicit Differentiation

Sometimes, we want to consider more generally equations involving x and y instead of functions $y=f(x)$.

Ex.: A circle of radius 1 is described by $x^2+y^2=1$.



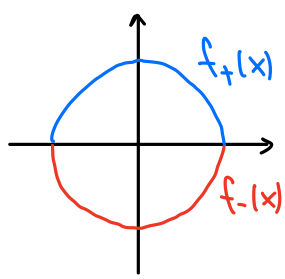
More precisely: The graph of $x^2+y^2=1$, i.e., all (x,y) satisfying the eq., is a circle.

In general, the graph of an eq. can be described in 3 different ways:

(1) An implicit eq. involving x and y.

Circle ex.: $x^2+y^2=1$

(2) Patch together the graphs of several functions.
Note that this is not always explicitly possible.



Circle ex.: $f_+(x)=\sqrt{1-x^2}$ and $f_-(x)=-\sqrt{1-x^2}$

(3) Parametrization: Graph $= \{(x,y)\in\mathbb{R}^2 : x = x(\varphi), y = y(\varphi), \varphi\in D\}$ for some $x(\varphi), y(\varphi)$ and domain $D\subset\mathbb{R}$.

Circle ex.: For $\varphi\in[0,2\pi]$, the circle is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

How do we find the derivative of a graph with those 3 possibilities?

(1) Implicit differentiation. We do not solve for $y = f(x)$ (this might not even be possible), but nevertheless we regard y as a fct. of x and differentiate the eq.

Circle ex.: We take the derivative on both sides of $x^2 + y^2 = 1$.

$$\Rightarrow \frac{d}{dx}(1) = 0 = \frac{d}{dx}(x^2+y^2) = 2x + \frac{d}{dx}\left(y(x)^2\right) \underset{\uparrow \text{ chain rule}}{=} 2x + 2y(x)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

(2) We compute the derivative for each patch.

Circle ex.: $\frac{dy}{dx} = \frac{d}{dx}\left(\pm\sqrt{1-x^2}\right) \underset{\uparrow \text{ chain rule}}{=} \pm\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\pm\sqrt{1-x^2}} = -\frac{x}{y}$

(3) Use the chain rule for the parametrization:

$$\frac{dy}{dx} = \frac{dy(\varphi)}{dx} \underset{\uparrow \text{ chain rule}}{=} \frac{dy(\varphi)}{d\varphi}\,\frac{d\varphi}{dx} \underset{\uparrow \text{ inverse fct. rule}}{=} \frac{dy(\varphi)}{d\varphi}\,\frac{1}{\frac{d(x(\varphi))}{d\varphi}}.$$

Circle ex.: $\frac{dy}{dx} = \frac{d\sin(\varphi)}{d\varphi}\,\frac{1}{\frac{d\cos(\varphi)}{d\varphi}} = \cos(\varphi)\,\frac{1}{-\sin(\varphi)} = -\frac{\cos(\varphi)}{\sin(\varphi)} = -\frac{x}{y}.$

Note: Implicit differentiation is usually the easiest.

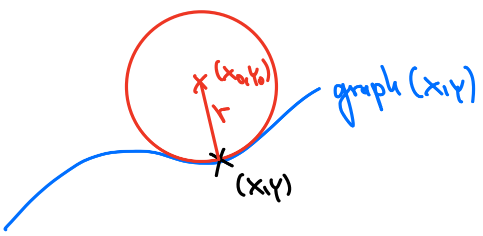
Another example: What is $\frac{dy}{dx}$ for $x^3 - 3xy + y^3 = 0$?

$$\Rightarrow \quad 0 = \frac{d}{dx}\left(x^3 - 3xy + y^3\right) = 3x^2 - 3\underbrace{\frac{d}{dx}(xy)}_{= 1y + x\frac{dy}{dx}} + 3y^2\frac{dy}{dx} = 3x^2 - 3y - 3x\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

$$= 3x^2 - 3y + \frac{dy}{dx}\left(-3x + 3y^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

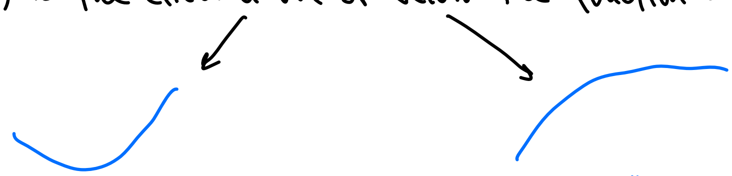
An application of implicit differentiation: Curvature and the osculating circle.



The osculating circle touches the graph at (x,y) and has the same first and second derivative at that point.

Questions: (1) What is the radius r of the circle?

(2) Is the circle above or below the function?



Called "concave up" or "convex". Called "concave down" or "concave".

The general eq. for a circle with radius r centered at (x_0, y_0) is $(x - x_0)^2 + (y - y_0)^2 = r$.

Implicit differentiation yields: $0 = \frac{d}{dx}\left((x-x_0)^2 + (y-y_0)^2\right) = 2(x-x_0) + 2(y-y_0)\frac{dy}{dx}$ (*)

Implicit differentiation again yields: $0 = \frac{d}{dx}\left(2(x-x_0) + 2(y-y_0)\frac{dy}{dx}\right)$

$$\overset{\text{product rule}}{=} 2 + 2\frac{dy}{dx}\frac{dy}{dx} + 2(y-y_0)\underbrace{\frac{d}{dx}\frac{dy}{dx}}_{=: \frac{d^2y}{dx^2} = (y'(x))' = y''(x)}$$

the second derivative

$$\Longrightarrow \quad y - y_0 = -\frac{1+(y'(x))^2}{y''(x)} \quad (*)$$

Then (*) yields: $x - x_0 = -(y-y_0)\,y'(x) \overset{(*)}{=} \frac{1+(y'(x))^2}{y''(x)}\,y'(x)$

$$\Longrightarrow r^2 = (x-x_0)^2 + (y-y_0)^2 = \left(\frac{1+y'(x)^2}{y''(x)}\,y'(x)\right)^2 + \left(\frac{1+y'(x)^2}{y''(x)}\right)^2$$
$$= \frac{(1+y'(x)^2)^3}{y''(x)^2}$$

We def. the curvature $\kappa := \frac{1}{r} = \frac{y''(x)}{(1+y'(x)^2)^{\frac{3}{2}}}$.

Note:
- If $y''(x) > 0$, then (*) implies $y < y_0$, hence the circle lies above the graph, i.e., it is "concave up".
- If $y''(x) < 0$, then (*) implies $y > y_0$, hence the circle lies below the graph, i.e., it is "concave down".