Elements of Calculus

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Lecture notes from Spring 2025

Example Session Week 4

Example Session for:

Week 4 A: Definition of Differentiation and Differentiation Rules

Week 4 B: Implicit Differentiation

Must-Know Derivatives

- Let us prove $(x^k)' = k x^{k-1}$ for $k \in \mathbb{N}$ by induction.

  Induction start: $k=1$: $x' = 1$ ✓

  Induction step: Assuming $(x^k)' = kx^{k-1}$, we check:

$$(x^{k+1})' = (x^k x)' \underset{\text{product rule}}{=} (x^k)' x + x^k x' = k x^{k-1} x + x^k = (k+1) x^{k+1} \quad ✓$$

- For $k \in \mathbb{N}$: $(x^{-k})' = \left(\frac{1}{x^k}\right)' \underset{\text{quotient rule}}{=} \frac{1' x^k - 1 (x^k)'}{x^{2k}} = \frac{-k x^{k-1}}{x^{2k}} = -k x^{-k-1}$

- For $p, q \in \mathbb{Z}$, $q \neq 0$: $(x^{\frac{p}{q}})' \underset{\text{chain rule}}{=} p (x^{\frac{1}{q}})^{p-1} (x^{\frac{1}{q}})'$

  inverse fct. rule: $f^{-1}(x) = y^{\frac{1}{q}} \Rightarrow x^q = y = f(x)$

$$\Rightarrow (y^{\frac{1}{q}})' = \frac{1}{f'(x)} = \frac{1}{q x^{q-1}} = \frac{1}{q y^{\frac{q-1}{q}}} = \frac{1}{q} y^{\frac{1}{q}-1}$$

$$\Rightarrow (x^{\frac{p}{q}})' = p x^{\frac{p-1}{q}} \frac{1}{q} x^{\frac{1}{q}-1} = \frac{p}{q} x^{\frac{p}{q}-1}$$

- $$(e^x)' = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x \underbrace{\lim_{h\to 0}\frac{\sum_{k=0}^{\infty}\frac{h^k}{k!}-1}{h}}_{=\lim_{h\to 0}\frac{h+\frac{h^2}{2}+\frac{h^3}{6}+\dots}{h}=1} = e^x$$

Or: $$(e^x)' = \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)' = \sum_{k=1}^{\infty}\frac{k}{k!}x^{k-1} = \sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$$

- $$(\ln x)' \underset{\uparrow}{=} \frac{1}{(e^y)'} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Inverse fct. rule, $x = e^y$

- For $r\in\mathbb{R}$: $x^r := e^{r\ln x}$

$$\Rightarrow (x^r)' = (e^{r\ln x})' = e^{r\ln x}(r\ln x)' = x^r\,\frac{r}{x} = r x^{r-1}$$

chain rule

- $$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \underset{\uparrow}{=} \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$\sin(x+h) = \sin x\cos h + \cos x \sin h$

$$= \sin(x)\lim_{h\to 0}\underbrace{\frac{\cos(h)-1}{h}}\ + \cos(x)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1} = \cos(x)$$

$$= \frac{-2\sin\left(\frac{h}{2}\right)^2}{h^2}\,h$$

$$= -\frac{1}{2}\underbrace{\frac{\sin\left(\frac{h}{2}\right)^2}{\left(\frac{h}{2}\right)^2}}_{\xrightarrow[h\to 0]{} 1}\ \underbrace{h}_{\xrightarrow[h\to 0]{} 0}$$

- $(\cos x)' = -\sin x$ follows similarly

- $f(x) = \tan x := \frac{\sin x}{\cos x}$

$$\Rightarrow f'(x) = \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{(\cos x)^2} \underset{(\sin x)^2 + (\cos x)^2 = 1}{=} \frac{1}{(\cos x)^2} = \frac{1}{\cos^2 x}$$

f^{-1} is called $\arctan y$, i.e., $\arctan(\tan x) = x$

$$\Rightarrow (\arctan y)' = \frac{1}{f'(x)} = \cos^2 x \text{ , with } y = \frac{\sin x}{\cos x}.$$

Need to solve for $\cos^2 x$ in terms of y: $y^2 = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} \Rightarrow \cos^2 x \, y^2 = 1 - \cos^2 x$

$$\Rightarrow \cos^2 x = \frac{1}{1+y^2}$$

$$\Rightarrow (\arctan y)' = \frac{1}{1+y^2}$$

More Examples of Derivatives

- $(\cos(x^2))' \underset{\text{chain rule}}{=} -\sin(x^2)\, 2x$

- $((x^3+4)^5)' \underset{\substack{\text{chain rule} \\ f(x)=x^5,\, g(x)=x^3+4}}{=} \underbrace{5(x^3+4)^4}_{f'(g(x))} \underbrace{3x^2}_{g'(x)} = 15x^2(x^3+4)^4$

- $\left(\frac{x^2+1}{x^5+x}\right)' \underset{\substack{\text{quotient} \\ \text{rule}}}{=} \frac{2x(x^5+x) - (x^2+1)(5x^4+1)}{(x^5+x)^2} = \frac{2x^6+2x^2-5x^6-x^2-5x^4-1}{(x^5+x)^2}$

$$= \frac{-3x^6-5x^4+x^2-1}{(x^5+x)^2}$$