Elements of Calculus

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Lecture notes from Spring 2025

Session 10
(Week 5B)

3. Differentiation in One Variable

3.2 Theorems and Applications

Topic for Week 5B: Critical Points, Inflection Points, and Graph Sketching

First, we discuss critical points such as minima, maxima, cusps. Then, we discuss qualitative features of graphs of functions.

Definition: Let $f: (a,b) \to \mathbb{R}$. Then any point $c \in (a,b)$ with $f'(c) = 0$ or where $f'(c)$ does not exist is called **critical point**.

Ex.: • $f(x) = |x|$ has a critical point at $c = 0$ since $f'(0)$ does not exist there.

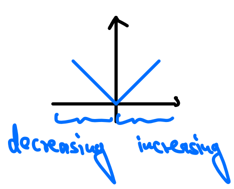
• $f(x) = x^2$ has a critical point at $c = 0$ since $f'(0) = 0$ there.

For both, $c = 0$ is a minimum.

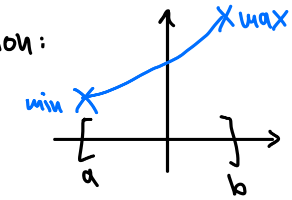
Note: • If f has a local extreme value (i.e., a max. or min.) at c, and is differentiable at c, then $f'(c) = 0$ (see Week 5A Session).

• If f is not differentiable at c, but is decreasing to the left and increasing to the right, it has a minimum. (Max. for increasing to the left and decreasing to the right.)

Ex.: $|x|$ at $x = 0$.



- Extreme values can also be endpoints of closed intervals of definition:



Sufficient conditions for the existence of an extreme value at x are:

- $f'(x)$ changes sign
- $f'(x)=0$, $f''(x)$ exists and is not zero (and continuous near x):
  - If $f''(x)>0$ then f' is increasing near x. Since $f'(x)=0$, f' must change sign from $-$ to $+$, so f has a min. at x.
  - Analogous: If $f''(x)<0$, then f has a max. at x.

Note: A point x where $f''(x)$ changes sign is called "point of inflection".

Example: $f(x)=\dfrac{x^2-1}{\sqrt{x^2+1}}$. Here, $f:\mathbb{R}\to\mathbb{R}$, so f is differentiable anywhere.

$$f'(x)=\frac{2x(x^2+1)^{\frac{1}{2}}-(x^2-1)\frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x}{x^2+1}$$

$$=\frac{2x(x^2+1)-x(x^2-1)}{(x^2+1)^{3/2}}$$

$$=\frac{x^3+3x}{(x^2+1)^{\frac{3}{2}}}$$

$$=\frac{x(x^2+3)}{(x^2+1)^{3/2}}$$ ← numerator $x(x^2+3)$ is >0 for $x>0$ and <0 for $x<0$
← denominator is >0 $\forall x\in\mathbb{R}$

$\Rightarrow f'(x)=0$ at $x=0$ and:
- $f'(x)>0$ for $x>0$ $\Rightarrow f$ is increasing for $x>0$
- $f'(x)<0$ for $x<0$ $\Rightarrow f$ is decreasing for $x<0$

$\Rightarrow f$ has a global minimum at $x=0$.

Check second derivative:

$$f''(x) = \left(\frac{x(x^2+3)}{(x^2+1)^{3/2}} \right)'$$

$$= \frac{[(x^2+3)+x2x](x^2+1)^{\frac{3}{2}} - \frac{3}{2}(x^2+1)^{\frac{1}{2}} 2x\, x(x^2+3)}{(x^2+1)^3}$$

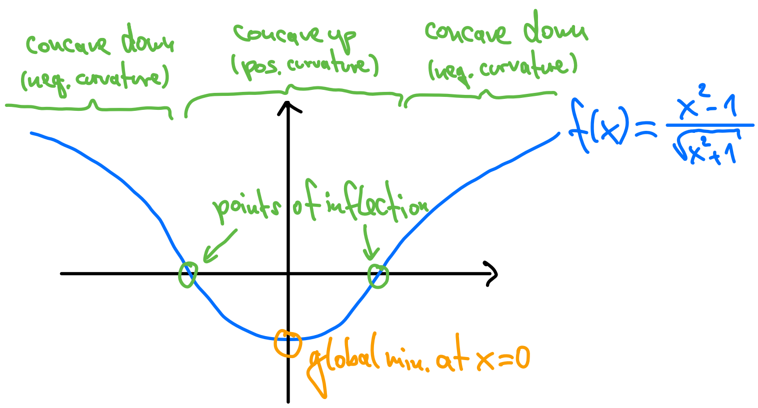
$$= \frac{(3x^2+3)(x^2+1) - 3(x^4+3x^2)}{(x^2+1)^{5/2}}$$

$$= \frac{-3x^2+3}{(x^2+1)^{5/2}}$$

$f''(0) = 3 > 0$, so we see again that the critical point $x=0$ is a minimum.

Moreover: $f''(x) = \frac{-3(x-1)(x+1)}{(x^2+3)^{5/2}}$ is changing signs at $x = \pm 1$

- At $x=-1$, f'' changes from $-$ to $+$
- At $x=1$, f'' changes from $+$ to $-$



In general, if we want to sketch the graph of a function with all qualitative features we have learnt about so far, we can proceed in the following way:

1. What is the domain of the function?
2. What are the y-intercepts ($f(0)$) and x-intercepts (solutions to $f(x)=0$)?
3. What are the horizontal and vertical asymptotes?
4. Discuss the first derivative: critical points and extrema, where is f increasing / decreasing?
5. Discuss the second derivative: points of inflection, where is f concave up / down?

We discuss examples in the Example Session.