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Lecture notes from Spring 2025

Example Session for:

Week 5 A: Theorems of Differentiation

Week 5 B: Critical Points, Inflection Points, and Graph Sketching

L'Hôpital

- For $\lambda > 0, n \in \mathbb{N}$: $\displaystyle \lim_{x\to\infty} \frac{e^{\lambda x}}{x^n} = \lim_{x\to\infty} \frac{\lambda e^{\lambda x}}{n x^{n-1}} = \ldots = \lim_{x\to\infty} \frac{\lambda^n e^{\lambda x}}{n!} \to \infty$

- For $\alpha > 0$: $\displaystyle \lim_{x\to\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x\to\infty} x^{-\alpha} = 0$

- $\displaystyle \lim_{x\to 0} x^x = \lim_{x\to 0} e^{x \ln x}$. We find: $\displaystyle \lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0.$

  e^x is cont.

  $= e^0 = 1$

- $\displaystyle \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$

Graph Sketching

Example 1: $f(x) = x^4 - 6x^3$

1. Domain is $D(f) = \mathbb{R}$

2. • y-intercept: $f(0) = 0 \Rightarrow x = y = 0$ is x and y-intercept
   • x-intercepts: $f(x) = x^3(x-6) = 0$ for $x = 0$ (as above) and $x = 6$.

3. • $f(x) \xrightarrow[x \to \pm\infty]{} \infty$, so no horizontal asymptotes.
   • No vertical asymptotes since $D(f) = \mathbb{R}$.

4. $f'(x) = 4x^3 - 18x^2 = 2x^2(2x - 9)$

   $\Rightarrow$ critical points are $x = 0$ and $x = \frac{9}{2}$.

   ↳ At $x_1 = 0$, f' does not change sign, so this is not an extrema

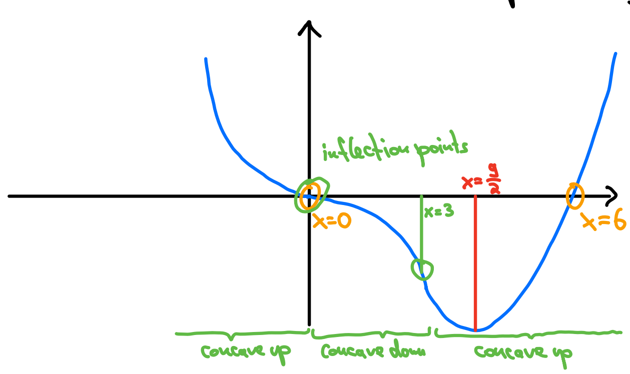
   ↳ For $x < \frac{9}{2}$: $f'(x) < 0$ (f decreasing)
   For $x > \frac{9}{2}$: $f'(x) > 0$ (f increasing)
   $\Rightarrow$ Minimum at $x_2 = \frac{9}{2}$.

5. $f''(x) = 12x^2 - 36x = 12x(x-3)$

   $\Rightarrow$ possible inflection points: $x = 0$, $x = 3$

   ↳ For $x < 0$: $f''(x) > 0$ (f concave up)
   ↳ For $0 < x < 3$: $f''(x) < 0$ (f concave down)
   ↳ For $x > 3$: $f''(x) > 0$ (f concave up)

   $\Big\}$ $x = 0$ and $x = 3$ are indeed inflection points



Example 2: $f(x)=\sqrt[3]{\frac{x^2}{(x-6)^2}} = \left(\frac{x^2}{(x-6)^2}\right)^{\frac{1}{3}}$

1. Domain $D(f)=\mathbb{R}\setminus\{6\}$

2. $f(0)=0 \Rightarrow (0,0)$ is x and y-intercept.

   $f(x)=0$ only at $x=0$, so there are no further x-intercepts.

3. • $\lim_{x\to\pm\infty} f(x) = 1 \Rightarrow y=1$ is horizontal asymptote for $x\to\pm\infty$

   • $\lim_{x\to 6} f(x)=\infty \Rightarrow x=6$ is vertical asymptote

4. $$f'(x)=\left(x^{\frac{2}{3}}(x-6)^{-\frac{2}{3}}\right)'$$
$$= \frac{2}{3}x^{-\frac{1}{3}}(x-6)^{-\frac{2}{3}} + x^{\frac{2}{3}}\left(-\frac{2}{3}\right)(x-6)^{-\frac{5}{3}}$$
$$= \frac{2}{3}x^{-\frac{1}{3}}(x-6)^{-\frac{5}{3}}\left((x-6)-x\right)$$
$$= -4x^{-\frac{1}{3}}(x-6)^{-\frac{5}{3}}$$

$\Rightarrow f'$ has no zeros. Also: Domain $D(f')=\mathbb{R}\setminus\{0,6\}$

$$\left.\begin{array}{l} \hookrightarrow \text{For } x<0:\ f'(x)<0 \quad (f \text{ decreasing}) \\ \hookrightarrow \text{For } 0<x<6:\ f'(x)>0 \quad (f \text{ increasing}) \end{array}\right\} f \text{ has a local minimum at } x=0.$$
$$\left.\begin{array}{l} \hookrightarrow \text{For } 0<x<6:\ f'(x)>0 \quad (f \text{ increasing}) \\ \hookrightarrow \text{For } x>6:\ f'(x)<0 \quad (f \text{ decreasing}) \end{array}\right\} \text{No max. at } x=6 \text{ since there is a vertical asymptote there}$$

5. $$f''(x)=-4\left(x^{-\frac{1}{3}}(x-6)^{-\frac{5}{3}}\right)'$$
$$=-4\left(-\frac{1}{3}x^{-\frac{4}{3}}(x-6)^{-\frac{5}{3}} + x^{-\frac{1}{3}}\left(-\frac{5}{3}\right)(x-6)^{-\frac{8}{3}}\right)$$
$$= \frac{4}{3}x^{-\frac{4}{3}}(x-6)^{-\frac{8}{3}}\left((x-6)+5x\right) = 8\underbrace{x^{-\frac{4}{3}}}_{\geq 0}\underbrace{(x-6)^{-\frac{8}{3}}}_{\geq 0}(x-1)$$

$\Rightarrow x = 1$ is a possible inflection point

↳ For $x < 1$: $f''(x) < 0$, so f is concave down

↳ For $x > 1$: $f''(x) > 0$, so f is concave up

} f has point of inflection at $x = 1$