Elements of Calculus — Session 11 (Week 6 A)

Prof. Sören Petrat, Constructor University

Lecture notes from Spring 2025

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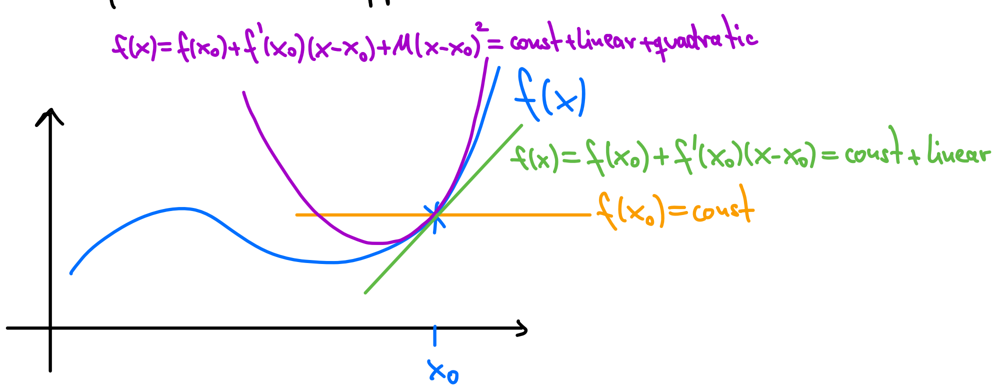
3. Differentiation in One Variable

3.2 Theorems and Applications

Topic for Week 6 A: Taylor Series

Goal: Go beyond the linear approximation of a function.



Idea: We approximate $f(x)$ near x_0 by a power series in $x - x_0$, i.e.: $f_T(x) = \sum_{k=0}^{N} c_k(x_0)(x-x_0)^k$.

Note:
- The more terms we take into account (i.e., the larger N), the better should be the approximation.
- The best approximation should hence be $N \to \infty$, i.e., $f_T(x) = \sum_{k=0}^{\infty} c_k(x_0)(x-x_0)^k$.
  Sometimes such a power series is even equal to the function, e.g., we know already that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ holds $\forall x \in \mathbb{R}$ (not just x near $x_0 = 0$).
- We know:
  - $f_T(x_0) = c_0(x_0) = f(x_0)$
  - $f_T'(x_0) = c_1(x_0) = f'(x_0)$

  The "zeroth" and first derivatives match.

We can find $c_2(x_0), c_3(x_0), \ldots$ by matching 2nd, 3rd, ... derivative at x_0:

- $\underbrace{f_T''}_{=f_T^{(2)}=\text{second derivative}}(x_0) = 2c_2(x_0) = f''(x_0) \implies c_2(x_0) = \frac{1}{2} f''(x_0)$

- $$f_T^{(3)}(x_0) = \left(\sum_{k=0}^{\infty} c_k(x_0)(x-x_0)^k\right)^{(3)}\Bigg|_{x=x_0}$$
$$= c_3(x_0)\left((x-x_0)^3\right)^{(3)}\Big|_{x=x_0}$$
$$= 3!\, c_3(x_0) = f^{(3)}(x_0) \implies c_3(x_0) = \frac{1}{3!} f^{(3)}(x_0)$$

- In general: If $f^{(k)}(x_0)$ denotes the k-th derivative at x_0, we find: $c_k(x_0) = \frac{1}{k!} f^{(k)}(x_0)$.

$$\implies f_T(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$, called Taylor series.

Next: Let us consider the error in the approximation

$$R_{N,x_0}(x) := f(x) - \underbrace{\sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{f_T(x)}$$, called remainder term.

Let us also define $S_{N,x_0}(x) := \frac{(x-x_0)^{N+1}}{(N+1)!}$.

We have shown above that the k-th derivative of the remainder at x_0 is zero for $k \le N$, i.e.,

$R_{N,x_0}^{(k)}(x_0) = 0$ for $k \le N$. Also: $R_{N,x_0}^{(N+1)}(x_0) = f^{(N+1)}(x_0)$.

For $S_{N,x_0}(x)$ we find:
- $S_{N,x_0}^{(k)}(x_0) = 0$ for $k \le N$
- $S_{N,x_0}^{(N+1)}(x_0) = 1$

Then Cauchy's theorem tells us that (assume $x_0 \le x$)

$$\frac{R_{N,x_0}(x)}{S_{N,x_0}(x)} = \frac{R_{N,x_0}(x) - \overbrace{R_{N,x_0}(x_0)}^{=0}}{S_{N,x_0}(x) - \underbrace{S_{N,x_0}(x_0)}_{=0}} \underset{\text{Cauchy}}{=} \frac{R'_{N,x_0}(m_1)}{S'_{N,x_0}(m_1)} \quad \text{for some } m_1 \in (x_0, x)$$

$$= \frac{R'_{N,x_0}(m_1) - \overbrace{R'_{N,x_0}(x_0)}^{=0}}{S'_{N,x_0}(m_1) - \underbrace{S'_{N,x_0}(x_0)}_{=0}} \underset{\text{Cauchy}}{=} \frac{R''_{N,x_0}(m_2)}{S''_{N,x_0}(m_2)} \quad \text{for some } m_2 \in (x_0, m_1)$$

$$= \dots = \frac{R^{(N+1)}_{N,x_0}(m_{N+1})}{S^{(N+1)}_{N,x_0}(m_{N+1})} = \frac{f^{(N+1)}(m_{N+1})}{1} \quad \text{for some } m_{N+1} \in (x_0, m_N).$$

$$\Rightarrow R_{N,x_0}(x) = S_{N,x_0}(x)\, f^{(N+1)}(m) = \frac{(x-x_0)^{N+1}}{(N+1)!} f^{(N+1)}(m) \quad \text{for some } m \in (x_0, x).$$

We have proven:

<u>Theorem (Taylor expansion):</u> Let f be continuous on $[x_0, x]$ and $(N+1)$ times differentiable in (x_0, x). Then $\exists\, m \in (x_0, x)$ s.t.

$$f(x) = \sum_{k=0}^{N} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \frac{(x-x_0)^{N+1}}{(N+1)!} f^{(N+1)}(m).$$

Note:
- $\sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$ is called **Taylor series of f at x_0**. If $x_0 = 0$, it is also called Maclaurin series.
- If f is arbitrarily often differentiable and $R_{N,x_0}(x) \xrightarrow{N\to\infty} 0$ near x_0, then $f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$ near x_0.
- In general: The Taylor series for arbitrarily often differentiable f might:
  - converge to f for some or all x,
  - converge, but not to f (if $R_{N,x_0}(x)$ does not converge to 0),
  - diverge (i.e., have convergence radius $\rho = 0$).

- We can also replace x by $x+x_0$ and write the Taylor series as $f(x+x_0)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}x^k$.

Examples:

- $f(x)=e^x \Rightarrow f^{(k)}(x)=e^x \Rightarrow f^{(k)}(0)=e^0=1$.

$\Rightarrow$ Taylor series of f around $x_0=0$ is $\sum_{k=0}^{\infty}\frac{x^k}{k!}$ and this equals e^x $\forall x\in\mathbb{R}$ (as we know from before).

- $f(x)=\sin x \Rightarrow$

$$\left.\begin{aligned} f'(x)&=\cos x\\ f''(x)&=-\sin x\\ f'''(x)&=-\cos x\\ f^{(4)}(x)&=\sin x \quad \text{and so on.}\end{aligned}\right\} \Rightarrow f^{(k)}(0)=\begin{cases}0, & k \text{ even}\\ (-1)^{\frac{k-1}{2}}, & k \text{ odd}\end{cases}$$

Remainder: $$|R_{N,0}(x)|=\left|\frac{x^{N+1}}{(N+1)!}\right|\underbrace{\left|f^{(N+1)}(u)\right|}_{\leq 1\ \forall u\in\mathbb{R}} \leq \underbrace{\left|\frac{x^{N+1}}{(N+1)!}\right| \xrightarrow{N\to\infty} 0}_{\text{Since } \sum_{k=0}^{\infty}\frac{x^k}{k!} \text{ bounded } \forall x\in\mathbb{R}.} \quad \forall x\in\mathbb{R}.$$

$$\Rightarrow \sin x=\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}(-1)^{\frac{k-1}{2}}\frac{x^k}{k!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{6}+\ldots$$

- Similarly: $$\cos x=\sum_{\substack{k=0\\ k\text{ even}}}^{\infty}(-1)^{\frac{k}{2}}\frac{x^k}{k!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2}+\ldots$$

- $$f(x)=e^{ix}=\sum_{k=0}^{\infty}\frac{(ix)^k}{k!}=\sum_{k=0}^{N} i^k\frac{x^k}{k!}=\underbrace{\sum_{\substack{k=0\\ k\text{ even}}}^{\infty}(-1)^{\frac{k}{2}}\frac{x^k}{k!}}_{\cos x}+i\underbrace{\sum_{\substack{k=1\\ k\text{ odd}}}^{\infty}(-1)^{\frac{k-1}{2}}\frac{x^k}{k!}}_{\sin x}$$

$$\Rightarrow \boxed{e^{ix}=\cos x+i\sin x \qquad \text{(Euler's formula)}}$$

More examples are discussed in the Homework exercises.