Elements of Calculus

Prof. Sören Petrat, Constructor University

Lecture notes from Spring 2025

Session 12 (Week 6B)

## 4. Integration in One Variable

**Topic for Week 6B: Indefinite Integrals**

"Integration = opposite to differentiation"

**Definition:**

Let $f, F : I \to \mathbb{R}$ ($I$ some interval), $F$ differentiable. Then $F$ is an antiderivative of $f$ if $F' = f$.

sometimes called "primitive"

Example: $f(x) = x^2 \Rightarrow F(x) = \frac{1}{3}x^3$ is an antiderivative, because $F'(x) = x^2$.

But note: $G(x) = \frac{1}{3}x^3 + 4$ is also an antiderivative.

We have:

**Theorem:**

$F$ and $G$ are antiderivatives of $f$ if and only if $G = F + c$ for some $c \in \mathbb{R}$.

**Proof:**

"$\Leftarrow$" If $G = F + c$, then $G' = (F+c)' = F' + \overbrace{c'}^{=0} = F'$, i.e., $F$ and $G$ are antiderivatives of the same function.

"$\Rightarrow$" Assume $F' = f$ and $G' = f$. Define $H = G - F$; then $H' = G' - F' = f - f = 0$.

By the mean-value theorem: $\frac{H(y) - H(x)}{y - x} = \underbrace{H'(z)}_{=0}$ for some $z \in (x, y)$.

$\Rightarrow H(y) = H(x)$ for $x \neq y$, i.e., $H(x) = c$ for some $c \in \mathbb{R}$. $\square$

__Definition:__

We call $\boxed{\underbrace{\int f(x)\,dx = F(x) + c}_{\text{here, } F \text{ is an antiderivative of } f} \text{ the indefinite integral}}$. Here, $c \in \mathbb{R}$ is called "constant of integration."

Examples:

- $\frac{d}{dx} x^{n+1} = (n+1)x^n$ for $n \in \mathbb{R}, n \neq -1$ $\Rightarrow$ $\boxed{\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c \quad , n \neq -1.}$

- $\frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$ $\Rightarrow$ $\boxed{\int \frac{1}{x}\,dx = \ln x + c \quad , x > 0.}$

  Note: for $x < 0$ we have $\frac{d}{dx} \ln(-x) = -\frac{1}{(-x)} = \frac{1}{x}$, so $\int \frac{1}{x}\,dx = \ln|x| + c$ on any interval not including $0$.

- $\frac{d}{dx} \cos x = -\sin x$, $\frac{d}{dx} \sin x = \cos x$ $\Rightarrow$ $\boxed{\int \sin x\,dx = -\cos x + c \text{ and } \int \cos x\,dx = \sin x + c.}$

- $\frac{d}{dx} e^x = e^x$ $\Rightarrow$ $\boxed{\int e^x\,dx = e^x + c.}$

- $\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$

  $\Rightarrow$ $\boxed{\int \frac{1}{\cos^2 x}\,dx = \tan x + c.}$

- $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ $\Rightarrow$ $\boxed{\int \frac{1}{1+x^2}\,dx = \arctan x + c.}$

  recall derivative of inverse function

Next: Must-know rules of integration which follow from key rules of differentiation.

- Product rule: $\frac{d}{dx}\left(F(x)g(x)\right) = F'(x)\,g(x) + F(x)\,g'(x)$

$$\Rightarrow F(x)g(x) = \int F'(x)g(x)\,dx + \int F(x)g'(x)\,dx$$

This is called "integration by parts": $\boxed{\int F'(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx.}$

Helpful when we know antiderivative of one fct. ($F'$) and when it helps to take the derivative of the other ($g$).

Examples:

- $\int \underset{F'}{e^x}\,\underset{g}{x}\,dx = \underset{F}{e^x}\,\underset{g}{x} - \int \underset{F}{e^x}\,\underset{g'}{1}\,dx = e^x x - e^x + c = e^x(x-1) + c.$

- $\int \underset{F'}{e^x}\,\underset{g}{\cos x}\,dx = \underset{F}{e^x}\,\underset{g}{\cos x} - \int \underset{F}{e^x}\,\underset{g'}{(-\sin x)}\,dx$

$$= e^x \cos x + \underbrace{\int e^x \sin x\,dx}$$

int. by parts again $\overset{?}{=} e^x \sin x - \int e^x \cos x\,dx$

$$\Rightarrow \int e^x \cos x\,dx = e^x(\cos x + \sin x) - \int e^x \cos x\,dx + c$$

$$\Rightarrow \int e^x \cos x\,dx = \frac{1}{2} e^x(\cos x + \sin x) + c.$$

- $\int \ln x\,dx = \int \underset{F'}{1} \cdot \underset{g}{\ln x}\,dx = \underset{F}{x}\,\underset{g}{\ln x} - \int \underset{F}{x}\,\underset{g'}{\frac{1}{x}}\,dx = x\ln x - x + C.$

- Chain rule: $\frac{d}{dx} F(u(x)) = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$ $\quad$ ($F' = f$ here).

This leads to $\boxed{\text{"integration by substitution": } \int f(u(x))\,u'(x)\,dx = F(u(x)) + c.}$

Useful for memorization (not rigorous): write $u'(x) = \frac{du}{dx} \Rightarrow u'(x)\,dx = du$

$$\Rightarrow \int f(u(x))\,u'(x)\,dx = \int f(u)\,du = F(u(x)) + c.$$

Examples:

- $\int \underbrace{e^x}_{u'(x)} \underbrace{\cos(e^x)}_{f(u(x))}\,dx = \int \cos(u)\,du = \sin(u(x)) + c = \sin(e^x) + c.$

  $(u(x) = e^x,\ u'(x) = e^x,\ f(y) = \cos(y))$

- $\int \underbrace{\sqrt{1+x^3}}_{f(u(x))}\ \underbrace{x^2}_{\frac{1}{3}u'(x)}\,dx = \int f(u(x))\,\frac{1}{3}u'(x)\,dx = \frac{1}{3}\int f(u)\,du = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\,\frac{2}{3}\,u(x)^{\frac{3}{2}} + c$

  $= \frac{2}{9}(1+x^3)^{\frac{3}{2}} + c.$

  $(u(x) = 1+x^3,\ u'(x) = 3x^2,\ f(y) = \sqrt{y})$

- $\int \tan x\,dx = \int \frac{1}{\cos x}\,\underbrace{\sin x\,dx}_{= u'(x)dx} \Rightarrow u(x) = -\cos x,\ f(y) = \frac{1}{y}$

  $= -\int \frac{1}{u}\,du = -\ln|u| + c,\ u \neq 0$

  $= -\ln|\cos(x)| + c$ $\quad$ (for intervals not including the zeroes of cos).