Elements of Calculus

Example Session Week 6

Prof. Sören Petrat, Constructor University

Lecture notes from Spring 2025

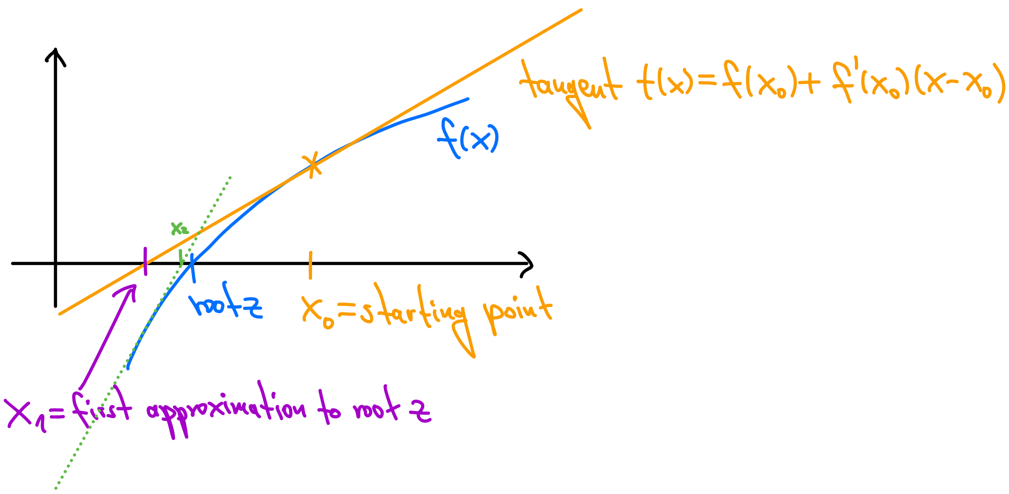
Example Session for:

Week 6 A: Taylor Series

Week 6 B: Indefinite Integrals

Newton's Method

Goal: Find roots, i.e., solutions to $f(x)=0$, by an iterative scheme using the derivative.

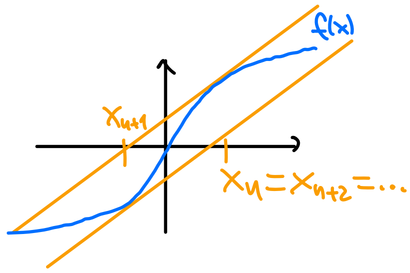


Setting $t(x_1)=0$ we find $0=f(x_0)+f'(x_0)(x_1-x_0) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

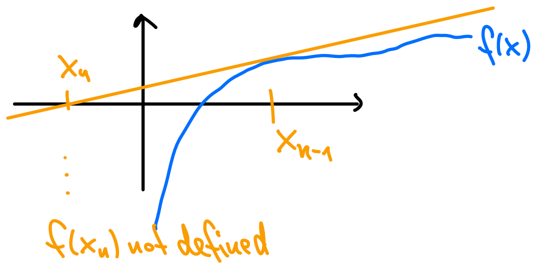
Repeating this yields the iteration $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ (Newton(-Raphson) method).

Note that this method might fail, i.e., x_n might not converge to a root, e.g., if

- $f'(x_n)=0$ for some x_n
- x_n might oscillate:



- $f(x_n)$ might not be defined:



Compared to that, the bisection method (see Week 3 Example Session) always works.

Ex.: $f(x)=x^2-a$ for $a \in \mathbb{R}$ $\Rightarrow$ $f'(x)=2x$

$\Rightarrow$ The iteration becomes $x_{n+1} = x_n - \frac{x_n^2-a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

E.g., for $a=2$, $x_0=1$, we find

- $x_1 = \frac{1}{2}\left(1+\frac{2}{1}\right) = \frac{3}{2}$
- $x_2 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2}\left(\frac{3}{2}+\frac{4}{3}\right) = \frac{17}{12} \approx 1.417\dots$, which is already close to $\sqrt{2} = 1.414\dots$

Next: How fast is the convergence to the root z (assuming the iteration does indeed converge)?

We write the iteration scheme as $x_{n+1} = F(x_n)$.

If z is a root, then $z = \lim_{n\to\infty} x_n$ and $\underbrace{z = F(z)}$.

This is called a "fixed point" (z does not change under application of F)

Let ε_n be the error between the approximation x_n and the real root z, i.e., $\varepsilon_n = x_n - z$.

$$\Rightarrow \quad x_{n+1} = \varepsilon_{n+1} + z = F(x_n) = F(z+\varepsilon_n) \qquad (*)$$

A Taylor expansion yields $F(z+\varepsilon_n) = \underbrace{F(z)}_{=z} + \varepsilon_n \underbrace{F'(z)} + \frac{\varepsilon_n^2}{2} F''(z) + \underbrace{R_2(z+\varepsilon_n)}$

We assume without proof that this is negligibly small.

$$F'(x) = \left(x - \frac{f(x)}{f'(x)}\right)' = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

quotient rule

$\Rightarrow F'(z) = 0$ since $f(z) = 0$ (z was def. as root of f).

Assuming $f'(z) \neq 0$.

$\Rightarrow$ $(*)$ yields $\quad \varepsilon_{n+1} + z = z + \frac{\varepsilon_n^2}{2} \underbrace{F''(z)}$

Another direct computation shows $F''(z) \neq 0$ if $f''(z) \neq 0$.

$\Rightarrow$ $\boxed{\varepsilon_{n+1} = \frac{F''(z)}{2} \varepsilon_n^2 \text{, which is called quadratic convergence}}$

(If Newton's method converges, and $f'(z) \neq 0$.)

Practical example: Suppose $F''(z) = 1$ and $\varepsilon_n = \frac{1}{10}$.

- linear convergence would mean $\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n = \frac{1}{2} \cdot \frac{1}{10} = \frac{1}{20}$, $\varepsilon_{n+2} = \frac{1}{2} \frac{1}{20} = \frac{1}{40}$, ...

  E.g., the bisection method converges linearly.

- quadratic convergence means $\varepsilon_{n+1} = \frac{1}{2}\left(\frac{1}{10}\right)^2 = \frac{1}{200}$, $\varepsilon_{n+2} = \frac{1}{2}\left(\frac{1}{200}\right)^2 = \frac{1}{80000}$, ...

  $\Rightarrow$ MUCH faster