Elements of Calculus

Session 13 (Week 7A)

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Lecture notes from Spring 2025

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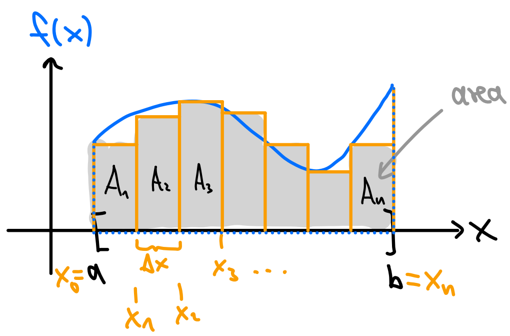
4. Integration in One Variable

Topic for Week 7A: Definite Integrals and the Fundamental Theorem of Calculus

So far: Indefinite integral was just a notation for antiderivative.

Now: Define the definite integral via limits and a geometrical meaning and then prove relationship to antiderivatives.

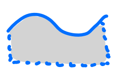
Idea: For $f:[a,b]\to\mathbb{R}$, we approximate the area between the graph of f and the x-axis.



$\Delta x = \frac{b-a}{n}$

$x_i = a + i\,\Delta x$

Approximation to area $= A = A_1 + A_2 + \dots + A_n = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

This should give the total area  as $n\to\infty$.

Definition:

Let $f:[a,b]\to\mathbb{R}$ be continuous except possibly at a countable number of points where f has jump discontinuities. Then the **definite integral** of f over $[a,b]$ is

$$\boxed{\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x.} \qquad \left(\Delta x = \frac{b-a}{n},\ x_i = a + i\,\Delta x\right)$$

Note: • The limit can be defined with arbitrary partitions of $[a,b]$, and can exist for a larger class of fct.s than stated. This is called "Riemann integral", and the rigorous construction is done in Analysis I.

• The fct. $f(x)=\begin{cases} 1 & \text{for } x\in\mathbb{Q}, \\ 0 & \text{for } x\in\mathbb{R}\setminus\mathbb{Q}, \end{cases}$ is an example of a fct. that is not integrable.

• For the stated class of fct.s the limit always exists. It is independent of the choice of evaluation of f in the intervals $[x_i, x_i+\Delta x]$, and of the choice of partition of $[a,b]$ as long as it gets finer.

Examples:

• $$\int_a^b 1\,dx = \lim_{n\to\infty}\sum_{i=1}^n 1\cdot\underbrace{\frac{b-a}{n}}_{=\Delta x} = b-a$$

• $$\int_a^b x\,dx = \lim_{n\to\infty}\sum_{i=1}^n x_{i-1}\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^n \left(a+(i-1)\left(\frac{b-a}{n}\right)\right)\frac{(b-a)}{n} = \lim_{n\to\infty}\left[\sum_{i=1}^n \frac{a(b-a)}{n} + \Big(\underbrace{\sum_{i=1}^n i}_{\overset{\text{e.g., by induction}}{=}\frac{n(n+1)}{2}} - \overbrace{\sum_{i=1}^n 1}^{=n}\Big)\left(\frac{b-a}{n}\right)^2\right]$$

$$= a(b-a) + \frac{(b-a)^2}{2}\underbrace{\lim_{n\to\infty}\frac{n(n-1)}{n^2}}_{=\lim_{n\to\infty}(1-\frac{1}{n})=1}$$

$$=\frac{1}{2}(b^2-a^2).$$

The following properties of the definite integral follow directly from the definition:

(i) Linearity: $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ and $\int_a^b c f(x)dx = c\int_a^b f(x)dx$, $c\in\mathbb{R}$,

(ii) $\int_a^a f(x)dx = 0$,

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,

(iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$. ← follows directly from (iii) with (ii)

Furthermore we have:

Theorem:

Let $f,g:[a,b]\to\mathbb{R}$ be integrable. Then:

(i) If $f\geq 0$, then $\int_a^b f(x)dx \geq 0$. (Clear from def.)

(ii) If $f\geq g$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$. (From (i) for $f-g$ instead of f.)

(iii) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$. (From (i) by splitting f into positive and negative part.)

(iv) If f is continuous, then there exists $z\in[a,b]$ s.t. $\underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{=\text{average of } f \text{ on } [a,b]} = f(z)$.

("Integral Mean-Value Theorem")

Proof of (iv): see exercise/question session.

Next: Connection to antiderivatives

Fundamental Theorem of Calculus (FTC):

Let $f:[a,b]\to\mathbb{R}$ be continuous. Then:

(i) $F(x)=\int_a^x f(t)dt$ is an antiderivative of f.

(ii) If F is an antiderivative of f, then $\boxed{\int_a^b f(x)dx = F(b)-F(a) =: F(x)\Big|_a^b.}$

Proof:

(i) We check: $\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right)$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Integral mean-value theorem

$$= f(z) \text{ for some } z\in[x,x+h]$$

$$\Rightarrow F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(z) \overset{f \text{ continuous}}{=} f(x).$$

(ii) We know $F(x) = \underbrace{\int_a^x f(t)\,dt}_{\text{This is an antiderivative according to (i).}} + c$ since antiderivatives are unique up to a constant.

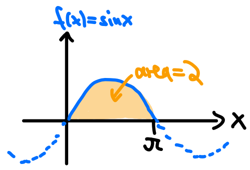
$$\Rightarrow F(a) = \underbrace{\int_a^a f(t)\,dt}_{=0} + c = c$$

$$\Rightarrow F(b) = \int_a^b f(t)\,dt + \overbrace{c}^{=F(a)}. \quad \checkmark \qquad \square$$

Examples:

- $\int_a^b x\,dx = \frac{1}{2}x^2\Big|_a^b = \frac{1}{2}(b^2-a^2)$ (as calculated above by hand)

- $\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi - (-\cos 0) = 1+1 = 2$



- $\int_0^1 \underbrace{x^2}_{\frac{u'(x)}{3}} \sin(\underbrace{x^3+1}_{u(x)})\,dx = \frac{1}{3}\int_{u(0)}^{u(1)} \sin(u)\,du$ $\qquad u(x) = x^3+1 \Rightarrow u'(x) = 3x^2$

$$= \frac{1}{3}\int_1^2 \sin(u)\,du$$

Limits of integration must be substituted as well!

$$= -\frac{1}{3}\cos u\Big|_1^2$$

$$= \frac{1}{3}(\cos(1)-\cos(2))$$

or $= \frac{1}{3}\int \sin u\,du$

$$= -\frac{1}{3}\cos u(x)\Big|_{x=0}^{x=1}$$

$$= -\frac{1}{3}\cos(x^3+1)\Big|_0^1$$

$$= \frac{1}{3}\left(\cos(1)-\cos(2)\right).$$

Note: By the FTC, $\frac{1}{b-a}\int_a^b f(x)\,dx \underset{\text{FTC}}{=} \frac{F(b)-F(a)}{b-a} \underset{\substack{\text{by mean value thm.}\\ \text{of differential calculus}}}{=} F'(z) = f(z)$ for some $z \in [a,b]$.

$\Rightarrow$ Mean-value thm.s of differential and integral calculus are related via the FTC.