Elements of Calculus Session 14 (Week 7B)

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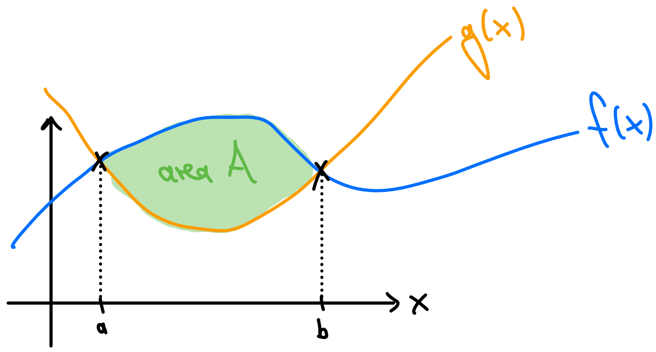
Lecture notes from Spring 2025

4. Integration in One Variable

Topic for Week 7B: Applications of Integration

Area between curves:

Compute area A between f and g:



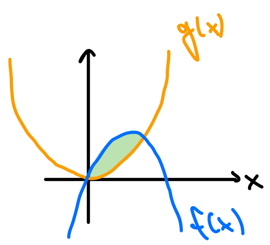
How? $\longrightarrow$ Find points of intersection.

$\longrightarrow$ Use integration to compute $\boxed{\int_a^b (f(x)-g(x))\,dx .}$

Example:

$g(x) = x^2$

$f(x) = 6x - 2x^2$



Points of intersection: $f(x) = g(x) \Rightarrow x^2 = 6x - 2x^2 \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow x(x-2) = 0$

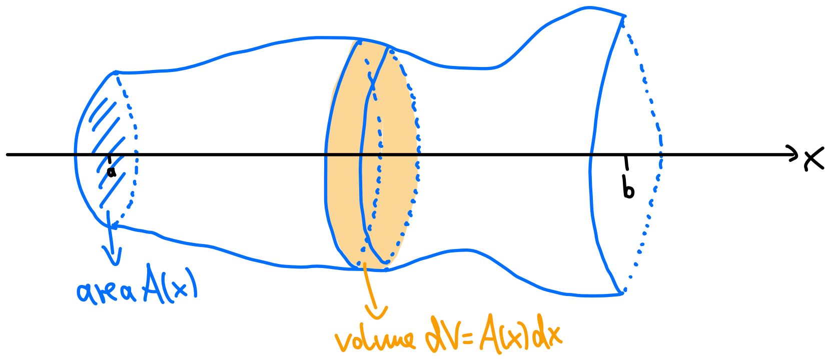
Points of intersection are at $x = 0 =: a$ and $x = 2 =: b$.

$$\Rightarrow \text{area } A = \int_0^2 (f(x)-g(x))\,dx = \int_0^2 (6x - 2x^2 - x^2)\,dx = \int_0^2 (-3x^2+6x)\,dx$$

$$= -x^3 + 3x^2 \Big|_0^2 = -8 + 12 - 0 = 4$$

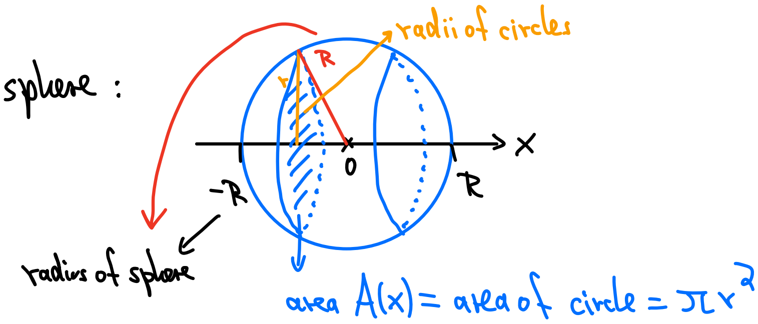
Volume computation:

Compute volume V if areas $A(x)$ of cross sections are known:



Here, volume $\boxed{V = \int_a^b \underbrace{A(x)\,dx}_{\text{"} = dV \text{"}}}$

Example: volume V of a sphere :



Note: $x^2 + r^2 = R^2 \Rightarrow A(x) = \pi r^2 = \pi(R^2 - x^2)$

$$\Rightarrow V = \int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \pi(R^2 - x^2)\,dx = \pi\left[R^2 x - \tfrac{1}{3}x^3\right]_{-R}^{R}$$

$$= \pi\left[R^3 - \frac{R^3}{3} - \left(-R^3 + \tfrac{1}{3}R^3\right)\right]$$

$$= \frac{4}{3}\pi R^3$$

Taylor Series

We reconsider the approximation of $f(x)$ near x_0 by a power series $f(x) = \sum_{k=0}^{N} c_k(x_0)\,(x-x_0)^k$.

By the FTC, we know: $\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0) \Rightarrow f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt$

Integration by parts: $\int_{x_0}^{x} \underbrace{1}_{=\frac{d}{dt}(t-x)} \cdot f'(t)\,dt = (t-x)\,f'(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} (t-x)\,f''(t)\,dt$

$$= (x-x_0)\,f'(x_0) + \int_{x_0}^{x} (x-t)\,f''(t)\,dt$$

$$\Rightarrow f(x) = f(x_0) + (x-x_0)\,f'(x_0) + \int_{x_0}^{x} (x-t)\,f''(t)\,dt$$

Another integration by parts yields:

$$f(x) = f(x_0) + (x-x_0)\,f'(x_0) + \frac{(x-x_0)^2}{2}\,f''(x_0) + \int_{x_0}^{x} \frac{(x-t)^2}{2}\,f'''(t)\,dt$$

In general, we recover the Taylor series we already know, but with a different (and often more useful) expression for the remainder term.

Theorem (Taylor expansion):
Let f be $(N+1)$ times continuously differentiable on $[x_0, x]$. Then

$$f(x) = \sum_{k=0}^{N} \frac{(x-x_0)^k}{k!}\,f^{(k)}(x_0) + \int_{x_0}^{x} \frac{(x-t)^N}{N!}\,f^{(N+1)}(t)\,dt.$$

Power Series

Let $f(x)=\sum_{k=0}^{\infty} a_k x^k$ have radius of convergence ρ.

Then:
- $f'(x)=\sum_{k=1}^{\infty} k a_k x^{k-1}$
- $\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$

Note:
- The radius of convergence is the same for $\sum a_k x^k$, $\sum k a_k x^{k-1}$, and $\sum \frac{a_k}{k+1} x^{k+1}$.
  This can be seen, e.g., from the ratio test.
- If $f(x)$ converges at $x=\pm\rho$, then $f'(x)$ does not necessarily converge there.
  (The extra k can make convergence harder.)
  Ex.: $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ has $\rho=1$, and the series converges at $x=1$. But the derivative $\sum_{k=1}^{\infty} \frac{1}{k} x^{k-1}$ does not converge at $x=1$.
- If $f(x)$ converges at $x=\pm\rho$, then so does $\int f(x)\,dx$.
  (The extra $\frac{1}{k+1}$ can make convergence better.)
  Ex.: Same as above.