Elements of Calculus

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Lecture notes from Spring 2025

Session 15
(Week 8 A)

4. Integration in One Variable

Topic for Week 8 A: Improper Integrals

Today we discuss the cases where

- the integration interval contains a singularity of the function, e.g., $\int_0^1 \frac{1}{x^\alpha} dx \quad (\alpha > 0)$,
- the integration interval is unbounded, e.g., $\int_0^\infty \frac{1}{1+x^2} dx$.

We define such integrals via limits.

Definition:

- Let $f : [a, \infty) \longrightarrow \mathbb{R}$ be integrable on $[a, r]$ for any $r > a$.

  Then $\boxed{\int_a^\infty f(x) dx := \lim_{r \to \infty} \int_a^r f(x) dx.}$ (Analogously: $\int_{-\infty}^a f(x) dx := \lim_{r \to -\infty} \int_r^a f(x) dx$.)

- Let $f : \mathbb{R} \to \mathbb{R}$ be integrable on any interval $[a, b]$.

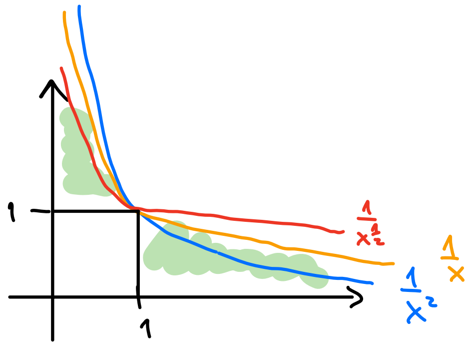
  Then $\boxed{\int_{-\infty}^\infty f(x) dx := \int_{-\infty}^0 f(x) dx + \int_0^\infty f(x) dx.}$ (Both limits need to exist!)

- Let $f : (a, b] \longrightarrow \mathbb{R}$ be integrable on $[r, b]$ for any $a < r < b$, and have a vertical asymptote at a.

  Then $\boxed{\int_a^b f(x) dx := \lim_{r \to a} \int_r^b f(x) dx}$.

These integrals are called **improper integrals** if the limits exist.

Examples:



- Let $\alpha \neq 1$: $\displaystyle\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{r\to\infty}\int_1^r \frac{1}{x^\alpha}\,dx = \lim_{r\to\infty} \frac{1}{1-\alpha} x^{-\alpha+1}\Big|_1^r = \frac{1}{1-\alpha}\lim_{r\to\infty}\left(r^{-\alpha+1}-1\right)$

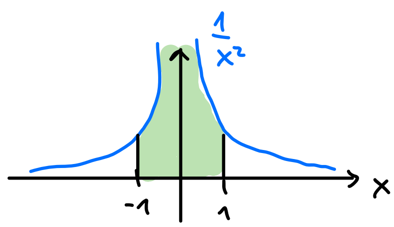
$$= \begin{cases} \frac{1}{\alpha-1} & \text{for } -\alpha+1<0 \text{, i.e., } \alpha>1, \\ \text{divergent} & \text{for } -\alpha+1>0 \text{, i.e., } \alpha<1. \end{cases}$$

At $\alpha = 1$: $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{r\to\infty} \ln x\Big|_1^r = \lim_{r\to\infty}(\ln r - 0) \longrightarrow \infty$

- Let $\alpha \neq 1$: $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha}\lim_{r\to 0}\left(1-r^{-\alpha+1}\right) = \begin{cases} \text{divergent} & \text{for } \alpha>1, \\ \frac{1}{1-\alpha} & \text{for } \alpha<1. \end{cases}$

At $\alpha = 1$: $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{r\to 0} \ln x\Big|_r^1 = \lim_{r\to 0}(0-\ln r) \longrightarrow \infty$

- $\cancel{\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^1 = -(1-(-1)) = -2}$???



$\frac{1}{x^2}$ has a vertical asymptote at $x=0$, so we need to split

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx = \lim_{r\to 0}\int_{-1}^r \frac{1}{x^2}dx + \lim_{r\to 0}\int_r^1 \frac{1}{x^2}dx = \lim_{r\to 0}\left[-\frac{1}{x}\right]_{-1}^r + \lim_{r\to 0}\left[-\frac{1}{x}\right]_r^1$$

$$= \infty$$

$\Rightarrow$ integral does not exist; area = infinite.

- $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{r\to\infty} \int_{-r}^{0} \frac{1}{1+x^2}\,dx + \lim_{r\to\infty} \int_{0}^{r} \frac{1}{1+x^2}\,dx$$
$$= \lim_{r\to\infty} \arctan x \Big|_{-r}^{0} + \lim_{r\to\infty} \arctan x \Big|_{0}^{r}$$
$$= \lim_{r\to\infty} \left(-\arctan(-r)\right) + \lim_{r\to\infty} \arctan r$$
$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$
$$= \pi$$

- For which values of p does the integral $\int_{e}^{\infty} \frac{1}{x(\ln x)^p}\,dx$ converge?

(We already know: divergent for $p=0$ (and thus $p<0$), but does $p>0$ help?)

$$\int_{e}^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_{1}^{\infty} \frac{1}{u^p}\,du = \begin{cases} \frac{1}{p-1} & \text{for } p>1, \\ \infty & \text{for } p\leq 1. \end{cases}$$

substitution: $u(x) = \ln x$
$\Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$
$u(e) = 1,\ u(\infty) = \infty$

$\Rightarrow$ Converges for $p>1$, diverges otherwise.

Applications:

- Integral test, see Homework: Let $f: [1,\infty) \to [0,\infty)$ be monotone decreasing ($f(y) < f(x)$ for $x>y$).
Then $\sum_{k=1}^{\infty} f(k)$ exists if and only if $\int_{1}^{\infty} f(x)\,dx$ exists.

- Interpolation of $n!$, see Homework: Def. the gamma function $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,dt$.
In the Homework we show that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

Example: Arrival times (e.g., in server queues).

Given arrival rate $\lambda > 0$, the time X between successive arrivals is often assumed to decay exponentially, i.e., as $f_\lambda(x) = \lambda \underbrace{e^{-\lambda x}}$.

high probability of job arriving soon, low probability of job arriving late.

To check that this is a probability distribution, we need to check that $\int_0^\infty f_\lambda(x)\,dx = 1$.

We compute: $\int_0^\infty \lambda e^{-\lambda x}\,dx = \lambda \left(\frac{1}{-\lambda}\right) e^{-\lambda x}\Big|_0^\infty = 0 + 1 = 1$ ✓

What is the average waiting time?

We compute: $\int_0^\infty x\, f_\lambda(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x}\,dx \overset{\text{int. by parts}}{=} x(-e^{-\lambda x})\Big|_0^\infty - \int_0^\infty (-e^{-\lambda x})\,dx$

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= \underbrace{\lim_{x\to\infty} (-xe^{-\lambda x})}_{=0 \text{ (L'Hospital)}} - 0 + \left[\frac{1}{-\lambda} e^{-\lambda x}\right]_0^\infty$$

$$= \frac{1}{\lambda}$$