Elements of Calculus

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Lecture notes from Spring 2025

Session 16
(Week 8B)

## 5. ODEs

**Topic for Week 8B: Ordinary Differential Equations**

In many applications, we are given a relation between functions and their derivatives, e.g., Newton's eq. $m \frac{d^2x(t)}{dt^2} = F(x(t))$.

Here, let us discuss the case where only the first derivative is involved.

**Definition:**

For some given function $f$, we call $\frac{dy}{dt} = f(y(t), t)$ a first-order **ordinary differential equation (ODE)**.

If $\frac{dy}{dt} = f(y(t))$ (no explicit $t$ dependence of $f$) we call the ODE **autonomous**.

If $\frac{dy}{dt} = f(y(t))\, g(t)$ for some fct. $g$ we call the ODE **separable**.

Goal: Find functions $y(t)$ that satisfy $\frac{dy}{dt} = f(y(t), t)$, for given $f$.

Given some initial time $t_0$ (often $t_0 = 0$), we call call $y_0 = y(t_0)$ the **initial condition**, which is given. Note that ODEs can have very different behavior depending on $y_0$.

A very important solution technique that works for separable ODEs is:

## Solution technique: $\boxed{\text{Separation of variables}}$

$$\boxed{\text{For } \frac{dy}{dt} = f(y)g(t), \text{ a solution can be found by integration: } \int_{y_0}^{y(t)} \frac{1}{f(y)}\,dy = \int_{t_0}^{t} g(x)\,dx.}$$

Note: • This works if $f$ and $g$ are continuous and $f(y) \neq 0$ for $y \in [y_0, y(t)]$.

$y_0 = y(t_0) =$ initial condition

• Formally: Bring all $y$'s to one side, all $t$'s to the other, and then integrate.

("$\frac{dy}{dt} = f(y)g(t) \Rightarrow \frac{dy}{f(y)} = g(t)\,dt \Rightarrow \int \frac{dy}{f(y)} = \int g(t)\,dt$")

Next: a few important examples

## Exponential Growth:

$\underbrace{\boxed{\frac{dy}{dt} = \lambda y}}_{\text{rate of change of } y \text{ is proportional to } y}$, with parameter $\lambda \in \mathbb{R}$ (separable and autonomous: $f(y) = \lambda y$, $g(t) = 1$)

rate coefficient for recovery

Examples: • epidemic: $\lambda = \beta - \gamma$, $y(0) = y_0 =$ initial number of infections

transmission rate coefficient

For $\lambda > 0$: number of infections is increasing,

$\lambda < 0$: number of infections is decreasing.

• radioactive decay: $\lambda < 0$

• generally: unrestricted population growth (for $\lambda > 0$) in this model

Solution: $\frac{dy}{dt} = \lambda y \Rightarrow \underbrace{\int_{y_0}^{y(t)} \frac{1}{y}\,dy}_{} = \lambda \underbrace{\int_0^t dt}_{= \lambda t}$

$= \ln y \Big|_{y_0}^{y(t)} = \ln y(t) - \ln y_0 = \ln \frac{y(t)}{y_0}$ (assume $y_0 > 0$)

$\Rightarrow \ln \frac{y(t)}{y_0} = \lambda t \Rightarrow \frac{y(t)}{y_0} = e^{\lambda t} \Rightarrow y(t) = y_0 e^{\lambda t}$ (with $y(0) = y_0$).

Note: Generally one (integration) constant (here $y_0$) is determined by initial conditions.

Note: Doubling time $T_2$ = time it takes for $y$ to double:

$$y(T_2) = y_0 e^{\lambda T_2} = 2y_0 \implies 2 = e^{\lambda T_2} \implies T_2 = \frac{\ln 2}{\lambda}.$$

(If $\lambda < 0$, we speak of "half-life", e.g., radioactive decay.)

## Limited Growth:

Growth might be limited, e.g., by limited food supply (e.g., fish in a pond).

$\implies$ Growth should stop once $y = k$ is reached, $k$ = "environmental carrying capacity".

$$\implies \boxed{\frac{dy}{dt} = \underbrace{\lambda y}_{\text{exp. growth}} \underbrace{\left(1 - \frac{y}{k}\right)}_{\text{stopped if } y=k}}$$

(this is called "logistics equation")

Separation of variables: $\displaystyle\int_{y_0}^{y(t)} \underbrace{\frac{1}{y(k-y)}} dy = \frac{\lambda}{k} \int_0^t dx = \frac{\lambda}{k} t$

recall integration of rational functions

$$= \frac{A}{y} + \frac{B}{k-y} = \frac{A(k-y) + By}{y(k-y)} \qquad \text{(partial fractions)}$$

$$\implies B - A = 0 \implies A = B \text{ and } Ak = 1 \implies A = B = \frac{1}{k}$$

$$\implies \int_{y_0}^{y(t)} \frac{1}{y(k-y)} dy = \frac{1}{k}\left(\int_{y_0}^{y(t)} \frac{1}{y} dy + \int_{y_0}^{y(t)} \frac{1}{k-y} dy\right)$$

$$= \frac{1}{k}\left(\ln y \Big|_{y_0}^{y(t)} - \ln(k-y)\Big|_{y_0}^{y(t)}\right) = \frac{1}{k} \ln \frac{y}{k-y}\Big|_{y_0}^{y(t)}$$

$$= \frac{1}{k}\left(\ln \frac{y(t)}{k-y(t)} - \ln \frac{y_0}{k-y_0}\right) = \frac{1}{k} \ln\left(\frac{y(t)}{k-y(t)} \frac{k-y_0}{y_0}\right)$$

$\Rightarrow$ Solution is given by $\ln\left(\frac{y(t)}{k-y(t)}\,\frac{k-y_0}{y_0}\right) = \lambda t$

$$\Rightarrow \frac{y(t)}{k-y(t)} = \frac{y_0}{k-y_0} e^{\lambda t} =: \alpha$$

Still need to solve $\frac{y}{k-y} = \alpha \Rightarrow y = \alpha(k-y) = \alpha k - \alpha y \Rightarrow (1+\alpha)y = \alpha k$

$$\Rightarrow y = \frac{\alpha}{1+\alpha} k = \frac{k}{\frac{1}{\alpha}+1}$$

$$\Rightarrow \boxed{y(t) = \frac{k}{\frac{(k-y_0)}{y_0}e^{-\lambda t}+1} = \frac{k y_0}{(k-y_0)e^{-\lambda t}+y_0}} \quad (\text{for } y_0 \geq 0)$$

Note: Initial condition: $y(t=0) = \frac{k y_0}{(k-y_0)+y_0} = y_0$ ✓

We could now plot $y(t)$ to discuss how it behaves for different initial data.

But even without an exact solution we can discuss the qualitative behavior.

Recall our ODE $\frac{dy}{dt} = \lambda y \left(1-\frac{y}{k}\right)$:

- for $y_0 = 0$: $\frac{dy}{dt} = 0 \Rightarrow y(t) = 0 \quad \forall t \geq 0$

  (check with exact solution: $y(t) = \frac{k \cdot 0}{(k-0)e^{-\lambda t}+0} = 0$ ✓)

- for $y_0 = k$: $\frac{dy}{dt} = 0 \Rightarrow y(t) = k \quad \forall t \geq 0$

  (check with exact solution: $y(t) = \frac{k \cdot k}{(k-k)e^{-\lambda t}+k} = k$ ✓)

These points where $\frac{dy}{dt} = 0$ are called $\boxed{\text{equilibrium points.}}$

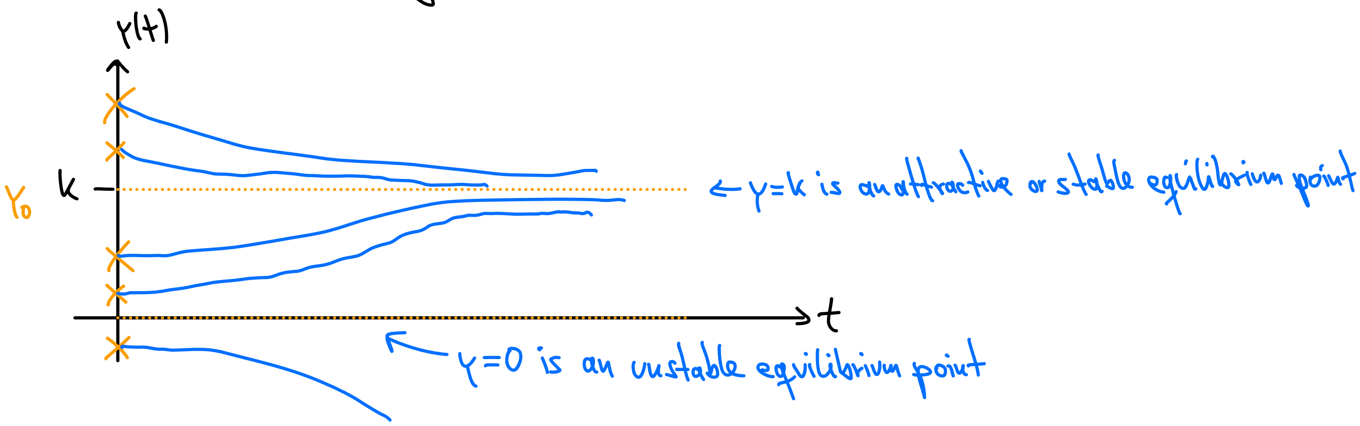
Then:

- when $y_0 > k$ : $\frac{dy}{dt} < 0$, i.e., $y$ is decreasing
- when $y_0 \in (0,k)$: $\frac{dy}{dt} > 0$, i.e., $y$ is increasing
- when $y_0 < 0$ : $\frac{dy}{dt} < 0$, i.e., $y$ is decreasing

for large $t$:

- $y(t) \to k$
- $y(t) \to k$
- $y(t) \to -\infty$

This yields the following qualitative sketch:




Note: Here, the solution is unique, so two solution curves cannot cross!