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Lecture notes from Spring 2025

Example Session for:

Week 8 A: Improper Integrals

Week 8 B: Ordinary Differential Equations

Queueing Theory (M|M|1)

The M|M|1 queue model (e.g., for jobs in a server) describes $N(t)$ = number of jobs in the system at time t given the parameters

- λ = job arrival rate
- μ = job processing rate

The model is $$\frac{dN(t)}{dt} = \underbrace{\lambda}_{\text{constant job arrivals}} - \overbrace{\mu N(t)}^{\text{jobs processed at rate } \mu}.$$

Separation of variables: $$\int_{N_0}^{N(t)} \frac{1}{\lambda - \mu N}\, dN = \overbrace{\int_0^t dt}^{=t}.$$

$$\int_{N_0}^{N(t)} \frac{1}{\lambda - \mu N}\, dN = \frac{1}{(-\mu)} \ln(\lambda - \mu N)\Big|_{N_0}^{N(t)} = -\frac{1}{\mu} \ln\left(\frac{\lambda - \mu N(t)}{\lambda - \mu N_0}\right)$$

$$\Rightarrow -\frac{1}{\mu} \ln \frac{\lambda - \mu N(t)}{\lambda - \mu N_0} = t \quad \Rightarrow \frac{\lambda - \mu N(t)}{\lambda - \mu N_0} = e^{-\mu t} \Rightarrow \lambda - \mu N(t) = (\lambda - \mu N_0) e^{-\mu t}$$

$$\Rightarrow N(t) = \frac{\lambda}{\mu} - \left(\frac{\lambda}{\mu} - N_0\right) e^{-\mu t} \qquad \left(\text{Double-check: } N(0) = \frac{\lambda}{\mu} - \left(\frac{\lambda}{\mu} - N_0\right) = N_0 \ \checkmark\right)$$

Note: $N(t) \xrightarrow[t \to \infty]{} \frac{\lambda}{\mu}$, i.e., the number of jobs reaches an equilibrium for large times.

Another Example of Separation of Variables:

$$\frac{dy}{dt} = -3yt \quad \text{with } y(0) = 1. \quad \text{(ODE is separable, but not autonomous.)}$$

$$\Rightarrow \underbrace{\int_1^{y(t)} \frac{1}{y}\, dy}_{\substack{= \ln y \,|_1^{y(t)} \\ = \ln y(t)}} = \underbrace{-3\int_0^t \tilde{t}\, d\tilde{t}}_{= -\frac{3}{2}t^2} \qquad \Rightarrow y(t) = e^{-\frac{3}{2}t^2}$$

Predator-Prey Models / Lotka-Volterra Equations:

A coupled system of two ODEs:

$$\text{prey: } \frac{dy}{dt} = \underbrace{by}_{\substack{\text{growth of} \\ \text{prey } y}} - \underbrace{rxy}_{\substack{\text{decline of population} \\ \text{through predators } x}} \qquad \text{predator: } \frac{dx}{dt} = \underbrace{-sx}_{\substack{\text{decline of predators} \\ x \text{ in absence of prey}}} + \overbrace{cxy}^{\text{growth of } x \text{ by availability of prey } y}$$

Here, we can deduce an equation relating y and x: $\frac{dy}{dx} = \frac{by - rxy}{-sx + cxy} = \frac{(b - rx)}{x} \frac{y}{(-s + cy)}$

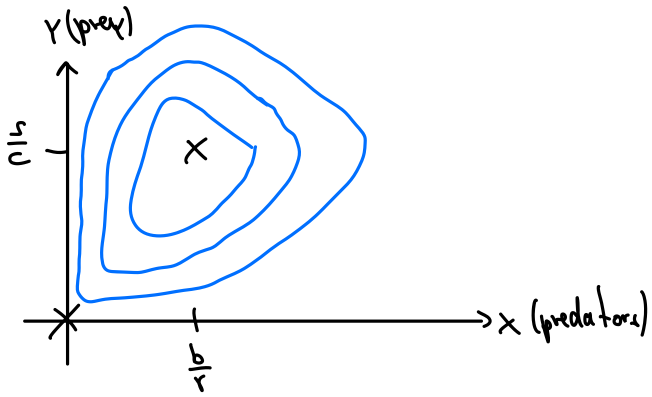
$$\Rightarrow \text{separation of variables: } \underbrace{\int \frac{-s+cy}{y}\, dy}_{\substack{= \int(-\frac{s}{y} + c)\, dy \\ = -s\ln y + cy}} = \underbrace{\int \frac{b - rx}{x}\, dx}_{= \int (\frac{b}{x} - r)\, dx = b\ln x - rx + \tilde{C}}$$

here, we don't specify the integration boundaries, but we introduce an integration constant $\tilde{C}$, which is determined by initial data

$$\Rightarrow -s\ln y + cy = b\ln x - rx + \tilde{C}$$

Note: Our ODE has two equilibrium points:
- $x = 0, y = 0$
- $x = \frac{b}{r}, y = \frac{s}{c}$

A plot of x vs. y yields:



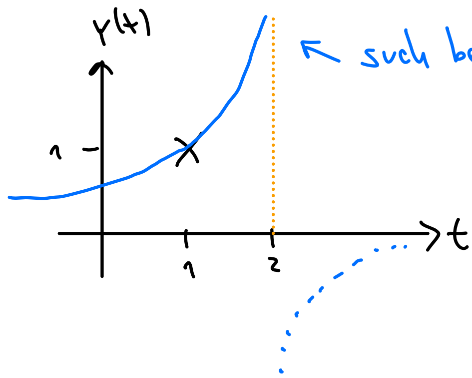
$\Rightarrow$ Populations are stable when $x = \frac{b}{r}$ and $y = \frac{s}{c}$, otherwise the populations grow/decline along one of the blue lines.

Blowup of Solutions

Consider $\frac{dy}{dt} = y^2$ with initial condition $y(1) = 1$.

$$\Rightarrow \int_1^{y(t)} \frac{1}{y^2}\, dy = \int_1^t dx \Rightarrow -\frac{1}{y}\Big|_1^{y(t)} = x\Big|_1^t \Rightarrow -\frac{1}{y(t)} + 1 = t - 1$$

$$\Rightarrow y(t) = \frac{1}{2-t}$$



such behavior is called "blow-up" at $t = 2$

Such blow-ups can appear if LHS of separation of variables has singularities (here, $\frac{1}{y^2}$).

Non-uniqueness of Solutions

Consider $\frac{dy}{dt} = \sqrt{y}$ with $y(0) = y_0$.

$$\Rightarrow \int_{y_0}^{y(t)} \frac{1}{\sqrt{y}}\, dy = \int_0^t dx \Rightarrow \quad 2y^{\frac{1}{2}} \Big|_{y_0}^{y(t)} = t \Rightarrow \sqrt{y(t)} = \frac{t}{2} + \sqrt{y_0}$$

$$\Rightarrow y(t) = \left(\frac{t}{2} + \sqrt{y_0}\right)^2$$

E.g., if $y_0 = 0$: $\quad y(t) = \frac{t^2}{4}$

But: $y(t) = 0$ is also a solution.

In fact, for any $c > 0$, $y(t) = \begin{cases} 0 & \text{for } t < c \\ \frac{(t-c)^2}{4} & \text{for } t \geq c \end{cases}$ is also a solution (for $y(0) = 0$)

So here the solution for the same initial condition is not unique.

(Reason is again the singularity of $\frac{1}{\sqrt{y}}$.)