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Lecture notes from Spring 2025

Example Session for:

Week 9 A: Finite Difference Methods

Week 9 B: Definitions of Total/Partial and Directional Derivatives

Finite Differences and Stability for the Damped Harmonic Oscillator

Consider the 2nd order ODE $\frac{d^2y}{dt^2} + 2\mu\omega\frac{dy}{dt} + \omega^2 y = 0$, with $\mu, \omega > 0$.

For $\mu = 0$ this is the harmonic oscillator, and $\mu > 0$ introduces damping.

By introducing the velocity $v = \frac{dy}{dt}$, we can reformulate the 2nd order ODE as two 1st order ODEs:

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2\mu\omega v - \omega^2 y$$

The forward Euler method yields:

- $\frac{y_{n+1} - y_n}{\Delta t} = v_n \implies y_{n+1} = y_n + \Delta t\, v_n$
- $\frac{v_{n+1} - v_n}{\Delta t} = -2\mu\omega v_n - \omega^2 y_n \implies v_{n+1} = v_n + \Delta t(-2\mu\omega v_n - \omega^2 y_n)$

In matrix/vector notation this reads
$$\begin{pmatrix} y_{n+1} \\ v_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \Delta t \\ -\omega^2 \Delta t & 1-2\mu\omega\Delta t \end{pmatrix}}_{\text{matrix } A} \begin{pmatrix} y_n \\ v_n \end{pmatrix}$$

For a stability analysis, we need to compute the eigenvalues of A. We find:

$$0 = \det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 1-\lambda & \Delta t \\ -\omega^2 \Delta t & 1-2\mu\omega\Delta t - \lambda \end{pmatrix} = (1-\lambda)(1-2\mu\omega\Delta t - \lambda) + \omega^2 (\Delta t)^2$$
$$= 1 - 2\mu\omega\Delta t - 2\lambda(1-\mu\omega\Delta t) + \lambda^2 + \omega^2 (\Delta t)^2$$
$$= \lambda^2 - 2\lambda(1-\mu\omega\Delta t) + 1 - 2\mu\omega\Delta t + \omega^2(\Delta t)^2$$

$$\Longrightarrow \lambda_\pm = 1 - \mu\omega\Delta t \pm \sqrt{\underbrace{(1-\mu\omega\Delta t)^2}_{1-2\mu\omega\Delta t + \mu^2\omega^2(\Delta t)^2} - 1 + 2\mu\omega\Delta t - \omega^2(\Delta t)^2}$$
$$= 1 - \mu\omega\Delta t \pm \sqrt{(\mu^2 - 1)\,\omega^2 (\Delta t)^2}$$
$$= 1 - \mu\omega\Delta t \pm \omega\Delta t \sqrt{\mu^2 - 1}$$
$$= 1 + \omega\Delta t \left(-\mu \pm \sqrt{\mu^2 - 1}\right)$$

The condition $|\lambda_\pm| < 1$ then leads to a stability condition.

Total | Partial | Directional Derivatives

Let us consider $f(x_1, x_2) = x_1^2 + x_2^2$.

- Total derivative:

We compute: $f(x_1+h_1, x_2+h_2) = (x_1+h_1)^2 + (x_2+h_2)^2 = x_1^2 + x_2^2 + 2x_1h_1 + 2x_2h_2 + h_1^2 + h_2^2$

$$= f(x_1, x_2) + 2(x_1, x_2)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \|h\|^2.$$

$\Rightarrow$ The total derivative of f is $Df|_x = 2(x_1, x_2)$.

- Partial derivatives: $\frac{\partial f}{\partial x_1} = 2x_1$, $\frac{\partial f}{\partial x_2} = 2x_2$

- Directional derivative: For $u \in \mathbb{R}^2$ with $\|u\| = \sqrt{u_1^2 + u_2^2} = 1$ we find

$$\frac{f(x+tu) - f(x)}{t} = \frac{(x_1+tu_1)^2 + (x_2+tu_2)^2 - (x_1^2 + x_2^2)}{t}$$

$$= \frac{2tu_1x_1 + t^2u_1^2 + 2tu_2x_2 + t^2u_2^2}{t}$$

$$= 2u_1x_1 + 2u_2x_2 + t(u_1^2 + u_2^2)$$

$$\xrightarrow[t \to 0]{} 2u_1x_1 + 2u_2x_2 = 2(x_1, x_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\Rightarrow D_u f|_x = 2(x_1, x_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$