Elements of Calculus

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Lecture notes from Spring 2025

Session 19
(Week 10 A)

6. Multivariable Calculus

6.1 Total and Partial Derivatives

Topic for Week 10 A: Connections between Total / Directional / Partial Derivatives

Recall the definitions of different derivatives for functions $f: \mathbb{R}^n \to \mathbb{R}^m$

(i.e., f takes in a vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and gives out a vector $\begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix}$).

- **Total derivative.** f is totally differentiable at x_0 if we can find an $m \times n$ matrix A s.t.

$$f(x_0 + h) = f(x_0) + Ah + r_{x_0}(h) \qquad \text{with} \quad \lim_{h \to 0} \frac{\|r_{x_0}(h)\|}{\|h\|} = 0.$$

If this is the case we call $A = Df|_{x_0}$ the total derivative of f at x_0.

- **Directional derivative.** We fix a direction $u \in \mathbb{R}^n$, $\|u\| = 1$. Then f is differentiable at x_0 in direction u if $\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$ exists. If it does, we define it to be the derivative of f at x_0 in direction u: $D_u f|_{x_0} = \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$

- **Partial derivative.** This is the special case of $u = e_j = j$-th Euclidean basis vector. f has a j-th partial derivative at x_0 if $\lim_{t \to 0} \frac{f(x_0 + te_j) - f(x_0)}{t}$ exist. In this case we call $\underbrace{\frac{\partial f}{\partial x_j}(x_0)}_{\text{sometimes we just write } \partial_j f(x_0)} = \lim_{t\to 0} \frac{f(x_0+te_j)-f(x_0)}{t}$ the j-th partial derivative of f at x_0.

Note: $Df|_{x_0}$ is an $m\times n$ matrix, $D_u f|_{x_0}$ is a vector in $\mathbb{R}^m$, $\frac{\partial f}{\partial x_j}(x_0)$ a vector in $\mathbb{R}^m$.

Example from last time: For $f(x_1,x_2) = \begin{pmatrix} x_1^2+x_1x_2 \\ 2x_1-x_2^2 \end{pmatrix}$ we find:

- $f(x+h) = f(x_1+h_1, x_2+h_2) = \underbrace{\begin{pmatrix} x_1^2+x_1x_2 \\ 2x_1-x_2^2 \end{pmatrix}}_{=f(x_1,x_2)} + \underbrace{\begin{pmatrix} 2x_1h_1+x_1h_2+x_2h_1 \\ 2h_1-2h_2x_2 \end{pmatrix}}_{=\begin{pmatrix} 2x_1+x_2 & x_1 \\ 2 & -2x_2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}} + \underbrace{\begin{pmatrix} h_1^2+h_1h_2 \\ -h_2^2 \end{pmatrix}}_{=r_x(h) \text{ with } \frac{\|r_x(h)\|}{\|h\|} \xrightarrow{h\to 0} 0}$

Hence the total derivative is $Df|_x = \begin{pmatrix} 2x_1+x_2 & x_1 \\ 2 & -2x_2 \end{pmatrix}$

- $$\lim_{t\to 0} \frac{f(x+tu)-f(x)}{t} = \lim_{t\to 0} \frac{1}{t}\left[\begin{pmatrix} (x_1+tu_1)^2+(x_1+tu_1)(x_2+tu_2) \\ 2(x_1+tu_1)-(x_2+tu_2)^2 \end{pmatrix} - \begin{pmatrix} x_1^2+x_1x_2 \\ 2x_1-x_2^2 \end{pmatrix}\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\begin{pmatrix} 2tu_1x_1+t^2u_1^2+tx_1u_2+tx_2u_1+t^2u_1u_2 \\ 2tu_1-2tx_2u_2-t^2u_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1u_1+x_1u_2+x_2u_1 \\ 2u_1-2x_2u_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1+x_2 & x_1 \\ 2 & -2x_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Hence, the derivative in direction u is $D_u f|_x = \begin{pmatrix} 2x_1+x_2 & x_1 \\ 2 & -2x_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

- The partial derivatives are:
  - $\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\begin{pmatrix} x_1^2+x_1x_2 \\ 2x_1-x_2^2 \end{pmatrix} = \begin{pmatrix} 2x_1+x_2 \\ 2 \end{pmatrix}$
  - $\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}\begin{pmatrix} x_1^2+x_1x_2 \\ 2x_1-x_2^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_2 \end{pmatrix}$

We could also see this by choosing $u=\begin{pmatrix}1\\0\end{pmatrix}$ for $\frac{\partial f}{\partial x_1}$ and $u=\begin{pmatrix}0\\1\end{pmatrix}$ for $\frac{\partial f}{\partial x_2}$.

We notice that in this example the derivatives are connected:

- $D_u f|_x = \underbrace{Df|_x \, u}_{\text{matrix times vector}}$

- $Df|_x = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}}_{\text{matrix with } \frac{\partial f}{\partial x_j} \text{ as column vectors}}$, which follows from $D_u f|_x = Df|_x \, u$ by choosing $u = e_j, j=1,\dots,n$.

The first equality holds not just in this example, but more generally.

Why? f differentiable at x_0 means $\lim\limits_{h\to 0} \frac{\|f(x_0+h)-f(x_0)-Df\,h\|}{\|h\|} = 0$.

In particular, for $u\in\mathbb{R}^n$, $\|u\|=1$, we can choose $h=tu$ and get

$$0 = \lim_{t\to 0} \frac{\|f(x_0+tu)-f(x_0)-Df(ut)\|}{t} = \lim_{t\to 0} \left\| \frac{f(x_0+tu)-f(x_0)}{t} - Df\,u \right\|$$

$$\text{i.e., } \lim_{t\to 0} \frac{f(x_0+tu)-f(x_0)}{t} = Df\,u.$$

Hence we have:

Theorem: If $f:\mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0\in\mathbb{R}^n$, then all directional derivatives at x_0 exist. In this case, the derivative in direction $u\in\mathbb{R}^n$, $\|u\|=1$, is given by

$$D_u f|_{x_0} = \underbrace{Df|_{x_0}}_{m\times n \text{ matrix}} \underbrace{u}_{\in\mathbb{R}^n}.$$

In particular, $\underbrace{\frac{\partial f_i(x)}{\partial x_j}}_{\substack{\text{derivative of the} \\ i\text{-th component of } f \\ \text{w.r.t. } x_j}} = \underbrace{(Df|_{x_0})_{ij}}_{\substack{(i,j) \text{ matrix entry of} \\ \text{the total derivative,} \\ = \text{the matrix of this} \\ \text{linear map in the basis } (e_j)}}$.

We call $\boxed{J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \text{ the Jacobian matrix}}$ of f at x.

But: There are examples of functions where all partial derivatives exist, but which are not differentiable (total derivative does not exist), e.g.,

- $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$ Here, the partial derivatives exist at $(0,0)$, but f is not even continuous there.

- $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$ Here, f is continuous at $(0,0)$ and all directional derivatives exist there. But f is not differentiable at $(0,0)$.

Geometrically, the problem is that we cannot put a tangent plane at the origin and get a good linear approximation with that. See the visualizations below, and see the Homework for proofs.

This cannot happen, however, if all partial derivatives are also continuous. (We skip the proof.)

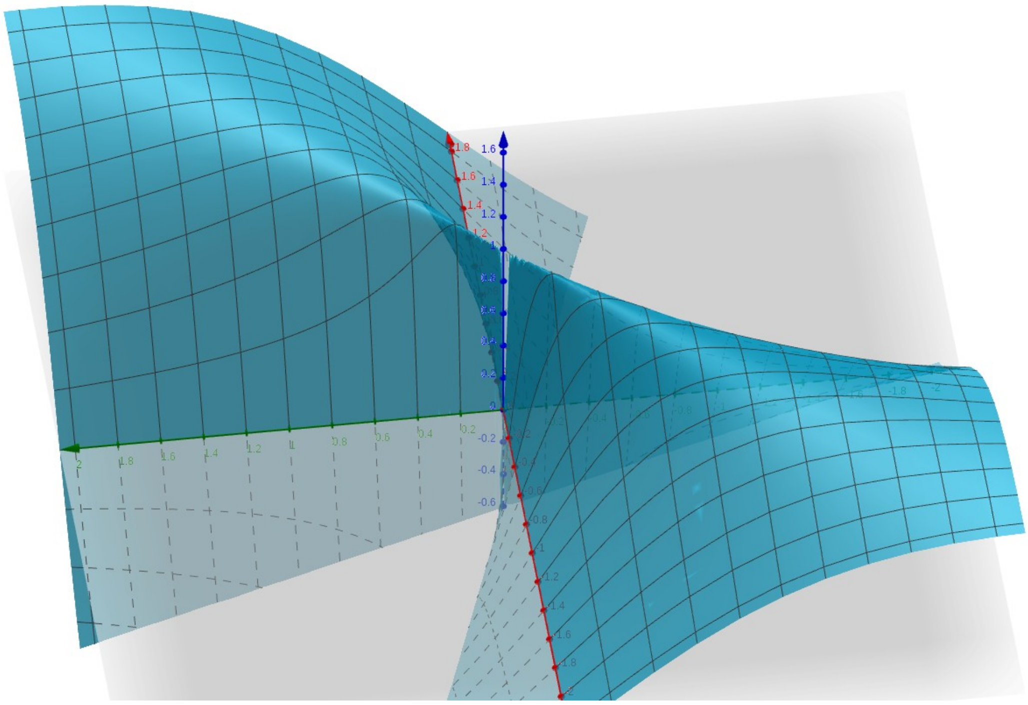
Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}^m$. Then f is totally continuously differentiable if and only if all partial derivatives exist and are continuous.

In the examples above, the partial derivatives exist, but they are not continuous.

So if all partial derivatives exist and are continuous, then Df = Jacobian.

See https://www.geogebra.org/3d for the plots.

$$f(x,y)=\frac{2xy}{x^2+y^2} \quad \text{for} \quad (x,y)\neq(0,0) \quad \text{and} \quad f(0,0)=(0,0)$$



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