Elements of Calculus

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Lecture notes from Spring 2025

Example Session Week 10

Example Session for:

Week 10 A: Connections between Total/Directional/Partial Derivatives

Week 10 B: Chain Rule, Gradient, Higher-order Derivatives, Taylor Expansion

Jacobian, Hessian, Taylor

Let us consider $f: \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \longmapsto f(x_1, x_2) = x_1 e^{x_1 x_2}$.

We find:

- The Jacobian is

$$Df|_x = \nabla f(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} f(x_1, x_2) = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(x_1 e^{x_1 x_2}\right) \\ \frac{\partial}{\partial x_2}\left(x_1 e^{x_1 x_2}\right) \end{pmatrix} = \begin{pmatrix} e^{x_1 x_2} + x_1 x_2 e^{x_1 x_2} \\ x_1^2 e^{x_1 x_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + x_1 x_2 \\ x_1^2 \end{pmatrix} e^{x_1 x_2}$$

- Furthermore: $$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left((1 + x_1 x_2) e^{x_1 x_2}\right) = x_2 e^{x_1 x_2} + (1 + x_1 x_2) x_2 e^{x_1 x_2}$$
$$= (2x_2 + x_1 x_2^2) e^{x_1 x_2}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}\left((1 + x_1 x_2) e^{x_1 x_2}\right) = x_1 e^{x_1 x_2} + (1 + x_1 x_2) x_1 e^{x_1 x_2}$$
$$= (2x_1 + x_1^2 x_2) e^{x_1 x_2}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(x_1^2 e^{x_1x_2}\right) = 2x_1 e^{x_1x_2} + x_1^2 x_2 e^{x_1x_2}$$
$$= (2x_1 + x_1^2 x_2) e^{x_1x_2}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(x_1^2 e^{x_1x_2}\right) = x_1^3 e^{x_1x_2}$$

Hence, the Hessian is $H_f(x) = \begin{pmatrix} 2x_2 + x_1x_2^2 & 2x_1 + x_1^2 x_2 \\ 2x_1 + x_1^2 x_2 & x_1^3 \end{pmatrix} e^{x_1x_2}$.

- Let us consider f near $x_0 = \binom{1}{1}$. We find:

$$f(1,1) = e$$

$$\nabla f|_{(1,1)} = \begin{pmatrix} 2e \\ e \end{pmatrix}$$

$$H_f|_{(1,1)} = \begin{pmatrix} 3e & 3e \\ 3e & e \end{pmatrix}$$

Hence, the second-order Taylor expansion of f around $(1,1)$ reads:

$$f\left(\binom{1}{1} + h\right) = f(1,1) + \langle \nabla f|_{(1,1)}, h\rangle + \frac{1}{2}\langle h, H_f|_{(1,1)} h\rangle + r(h)$$

$$= e + (2e, e)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \frac{1}{2}(h_1, h_2)\begin{pmatrix} 3e & 3e \\ 3e & e \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + r(h)$$

Leibniz integral rule

Consider $I(x) = \int_a^b f(x,t)\,dt = F(x,b) - F(x,a)$ $\left(\text{where } \underbrace{\frac{\partial F}{\partial t} = f}\right).$

keep x fixed

i.e. F is the antiderivative of f w.r.t. the t variable

Then
$$\frac{dI(x)}{dx} = \frac{\partial F(x,b)}{\partial x} - \frac{\partial F(x,a)}{\partial x}$$

FTC
$$= \int_a^b \frac{\partial}{\partial t}\left(\frac{\partial F(x,t)}{\partial x}\right) dt$$

Clairaut/Schwarz
$$= \int_a^b \frac{\partial}{\partial x} \underbrace{\frac{\partial F(x,t)}{\partial t}}_{= f(x,t)} dt$$

$$\Rightarrow \quad \boxed{\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt}$$

Without proof, we note the following:

Theorem: This formula is true if $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ are continuous.

Example: $I(x) = \int_a^b \frac{\sin(xt)}{t}\,dt$, for $0 < a < b$ (s.t. conditions of theorem hold).

$I(x)$ is not so easy to integrate. But we can easily compute:

$$\frac{dI(x)}{dx} = \int_a^b \frac{\partial}{\partial x}\left(\frac{\sin(xt)}{t}\right) dt = \int_a^b \cos(tx)\,dt = \left.\frac{\sin(tx)}{x}\right|_a^b = \frac{\sin(bx)}{x} - \frac{\sin(ax)}{x}$$