Elements of Calculus

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Lecture notes from Spring 2025

Session 21
(Week 11 A)

6. Multivariable Calculus

6.1 Total and Partial Derivatives

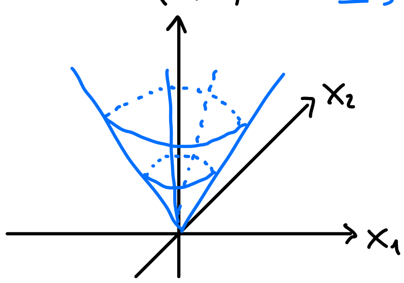
Topic for Week 11 A: Change of Variables, Differentials, Differential Operators

We continue with more remarks on multivariable Calculus.

Change of Variables

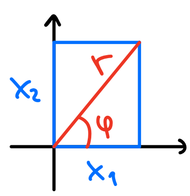
Sometimes we would like to express functions in terms of different variables (or "coordinates"), e.g. when they have a certain symmetry.

Ex.: $f(x_1,x_2) = \sqrt{x_1^2+x_2^2}$



$\Rightarrow$ This fct. is radially symmetric

Let us discuss the example of polar coordinates. Instead of expressing a fct. in terms of x_1, x_2, let us express it in terms of radius r and angle φ:



$\Rightarrow x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, with $r\in[0,\infty)$, $\varphi\in[0,2\pi)$

Ex.: What is the derivative in radial direction of $f(x_1,x_2)=\sqrt{x_1^2+x_2^2}$?

The chain rule tells us that $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial r}$, but instead of computing this we note that $f(x_1(r,\varphi), x_2(r,\varphi)) = f(r,\varphi) = \sqrt{(r\cos\varphi)^2+(r\sin\varphi)^2} = r$

$$\Rightarrow \frac{\partial f}{\partial r} = 1.$$

Next: What are $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ in terms of r,φ?

Note that $r(x_1,x_2)=\sqrt{x_1^2+x_2^2}$, $\varphi(x_1,x_2)=\arctan\left(\frac{x_2}{x_1}\right)$.

The chain rule tells us that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x_1}$ and $\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x_2}$

We can then compute: $\frac{\partial r}{\partial x_1} = 2x_1 \frac{1}{2}\left(x_1^2+x_2^2\right)^{-\frac{1}{2}} = \frac{r\cos\varphi}{r} = \cos\varphi$

$$\frac{\partial \varphi}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\arctan\left(\frac{x_2}{x_1}\right)\right) = \dots = -\frac{\sin\varphi}{r}$$

$$\Rightarrow \frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi} \quad \text{and similarly} \quad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}$$

Differentials

Consider one variable fct.s f and V that are related by $f(x)=\frac{dV(x)}{dx}$.

Here:
- Given V, we can directly compute f.
- Given f, we need to integrate to find V. In "separation of variables notation," we have

$$\int_{V(0)}^{V(\tilde{x})} dV = \int_0^{\tilde{x}} f(x)dx.$$ This is often written as a differential: $dV = f\,dx$

In many variables the situation is more complicated. Suppose $F:\mathbb{R}^n\to\mathbb{R}^n$ and $V:\mathbb{R}^n\to\mathbb{R}$ are related by $F=\nabla V$.

Then: • Given V, we can still directly compute F.

• Given F, can we find a V?

$\frac{\partial V}{\partial x_i} = F_i$

In differential notation: $dV = \frac{\partial V}{\partial x_1} dx_1 + \dots + \frac{\partial V}{\partial x_n} dx_n = F_1 dx_1 + \dots + F_n dx_n$

If such a V exists, we call dV an **exact differential**, if not an **inexact differential**.

Ex.: $n=2$, then $dV = \underbrace{F_1(x_1,x_2)}_{= \frac{\partial V}{\partial x_1}} dx_1 + \underbrace{F_2(x_1,x_2)}_{= \frac{\partial V}{\partial x_2}} dx_2$

$$\Rightarrow \frac{\partial F_2}{\partial x_1} = \frac{\partial}{\partial x_1} \frac{\partial V}{\partial x_2} \quad \text{and} \quad \frac{\partial F_1}{\partial x_2} = \frac{\partial}{\partial x_2} \frac{\partial V}{\partial x_1}$$

But these are equal by Clairaut/Schwarz

$\Rightarrow$ Need $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$ as a necessary condition.

It turns out that this condition is also sufficient. In general, it holds that

$$\boxed{dV = \sum_{i=1}^{n} F_i(x_1,\dots,x_n)\, dx_i \text{ is exact} \iff \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall i,j = 1,\dots,n}$$

Differential Operators

Certain common operations involving partial derivatives have names.

As before, we use $\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$ as a vector that can operate on functions in different ways:

• For $V: \mathbb{R}^n \to \mathbb{R}$ (a scalar field), $\nabla V = \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{pmatrix} =: \operatorname{grad} V$ is called **gradient of V**.

- For $f: \mathbb{R}^n \to \mathbb{R}^n$ (a vector field), $\underbrace{\nabla \cdot f}_{\text{scalar product}} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} =: \operatorname{div} f$

  is called $\boxed{\text{divergence of } f.}$

- For $f: \mathbb{R}^3 \to \mathbb{R}^3$ (a vector field), $\underbrace{\nabla \times f}_{\text{cross product}} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{pmatrix} =: \operatorname{curl} f$

  is called $\boxed{\text{curl of } f.}$

These operations get really interesting in the context of line/surface/volume integrals.
Just as a preview:

- Gauss's theorem says that $\int_V \operatorname{div} f \underbrace{dV}_{\substack{\text{volume} \\ \text{integral}}} = \int_{\partial V} f \cdot \hat{n} \underbrace{dS}_{\text{surface integral}}$

  $\hat{n}$ = normal vector of surface ∂V enclosing the volume V

- Stokes's theorem says that $\int_S (\operatorname{curl}(f)) \cdot \hat{n} \underbrace{dS}_{\substack{\text{surface} \\ \text{integral}}} = \int_{\partial S} f \cdot \underbrace{dx}_{\substack{\text{curve integral along the} \\ \text{boundary of the surface}}}$

These are the appropriate generalizations of the FTC in higher dimensions.