Elements of Calculus

Prof. Sören Petrat, Constructor University

Lecture notes from Spring 2025

Session 22 (Week 11B)

6. Multivariable Calculus

6.2 Optimization in Many Variables

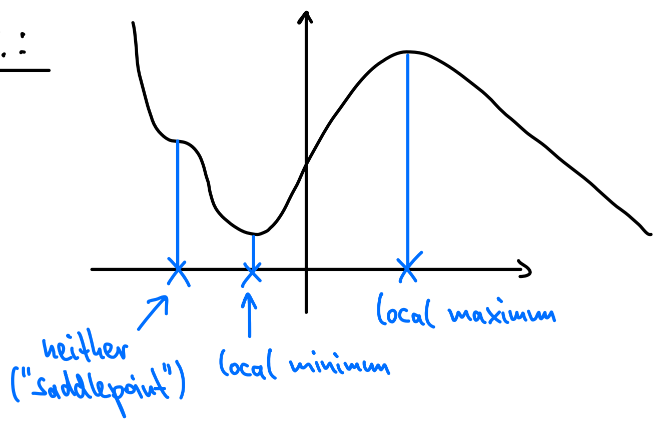
Topic for Week 11B: Critical Points, Maxima and Minima

Let us recall the following way of finding maxima/minima in one-variable Calculus:

Let $f: \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then c is called a critical (or stationary) point if $\frac{df}{dx}(c) = 0$. Furthermore:

- If $\frac{d^2f}{dx^2}(c) > 0$, then c is a local minimum
- If $\frac{d^2f}{dx^2}(c) < 0$, then c is a local maximum
- If $\frac{d^2f}{dx^2}(c) = 0$, then the 2nd derivative test is inconclusive (can be max or min or neither)

Ex.:



Today: What about max/min of $f:\mathbb{R}^n \to \mathbb{R}$?

As in $\mathbb{R}$, we say that f has a local maximum (minimum) at $c \in \mathbb{R}^n$ if $f(x) \leq f(c)$ ($f(x) \geq f(c)$) for all x near c.

Also recall from Session 10 B:

If f has a local max. or min. at $c \in \mathbb{R}^n$, then $\nabla f(c) = 0$.

(This can nicely be seen by the chain rule: For any direction $u \in \mathbb{R}^n$, $\|u\| = 1$, the function $g(t) = f(c + tu)$ has a local max./min. at $t = 0$ $\Rightarrow$ $0 = \frac{dg}{dt}(0) \underset{\text{chain rule}}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(c) \frac{\partial (c_i + t u_i)}{\partial t} = \nabla f(c) \cdot u$ $\Rightarrow \nabla f(c) = 0$.)

Examples (see geogebra pictures below):

- $f(x_1, x_2) = -x_1^2 - x_2^2$ has a maximum at $\binom{0}{0}$.
- $f(x_1, x_2) = x_1^2 - x_2^2$ has a saddlepoint at $\binom{0}{0}$, i.e., a max. in one direction, a min. in another direction.
- $f(x_1, x_2) = x_1^3 - x_2^2$ has neither a max., nor a min., nor a saddlepoint at $\binom{0}{0}$. We call this a degenerate critical point.

$\Rightarrow$ A critical point can be a local max, local min, a saddle point, or neither.

For a second derivative test in $\mathbb{R}^n$, recall the Taylor expansion for $f: \mathbb{R}^n \to \mathbb{R}$:

$$f(c+h) = f(c) + \underbrace{\nabla f(c)}_{\substack{=0 \text{ if } c \text{ is} \\ \text{a critical point}}} \cdot h + \frac{1}{2} \langle h, H_f(c) h \rangle + r_c(h)$$

, with $H_f(c)$ the Hessian matrix with entries $(H_f(c))_{ij} = \frac{\partial^2 f(c)}{\partial x_i \partial x_j}$.

Hence: • If $\langle h, H_f(c)h \rangle > 0$ for all h, c is a local min
• If $\langle h, H_f(c)h \rangle < 0$ for all h, c is a local max
• If $\langle h, H_f(c)h \rangle > 0$ for some h and < 0 for all others, c is a saddle point
• If $\langle h, H_f(c)h \rangle = 0$ for some h, we need to look at higher derivatives.

Now recall from Elements of Linear Algebra that a matrix H with $\langle h, Hh \rangle > 0 \ \forall h \in \mathbb{R}^n$ is called positive definite. And a real symmetric matrix is positive definite if and only if all eigenvalues are positive. (See Week 10 A Session of Elements of Linear Algebra.)

Everything works (H_f is symmetric, $r_c(h)$ is small) if all second partial derivatives of f are continuous, hence we can formulate:

<u>Theorem:</u> Let $f: \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable, and suppose $Df(c) = 0$ for some $c \in \mathbb{R}^n$. Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the Hessian of f at c. Then:
• If all $\lambda_i > 0$, then c is a local minimum.
• If all $\lambda_i < 0$, then c is a local maximum.
• If some $\lambda_i > 0$ and the other $\lambda_i < 0$ (but none equal 0), then c is a saddle point.
• If at least one $\lambda_i = 0$, then the test is inconclusive.

Examples:

• $f(x,y) = -x^2 - y^2 \Rightarrow Df = \begin{pmatrix} -2x \\ -2y \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is critical point

$$H_f(0) = \begin{pmatrix} (\partial_x^2 f)(0) & (\partial_x \partial_y f)(0) \\ (\partial_x \partial_y f)(0) & (\partial_y^2 f)(0) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \text{local max.}$$

- $f(x,y) = x^2 - y^2$

$$H_f(0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \text{saddle point}$$

- $f(x,y) = y^3 - 3x^2y$

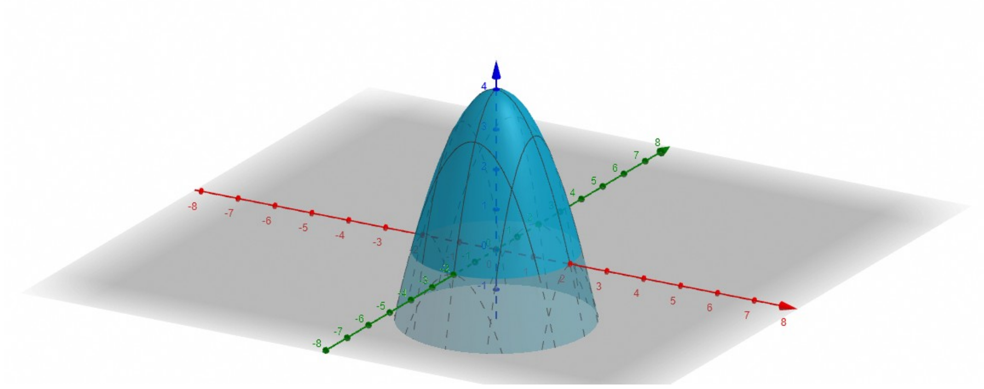
$$(\nabla f) = \begin{pmatrix} -6xy \\ 3y^2 - 3x^2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is the only critical point}$$

$$H_f = \begin{pmatrix} -6y & -6x \\ -6x & 6y \end{pmatrix} \Rightarrow H_f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{inconclusive}$$

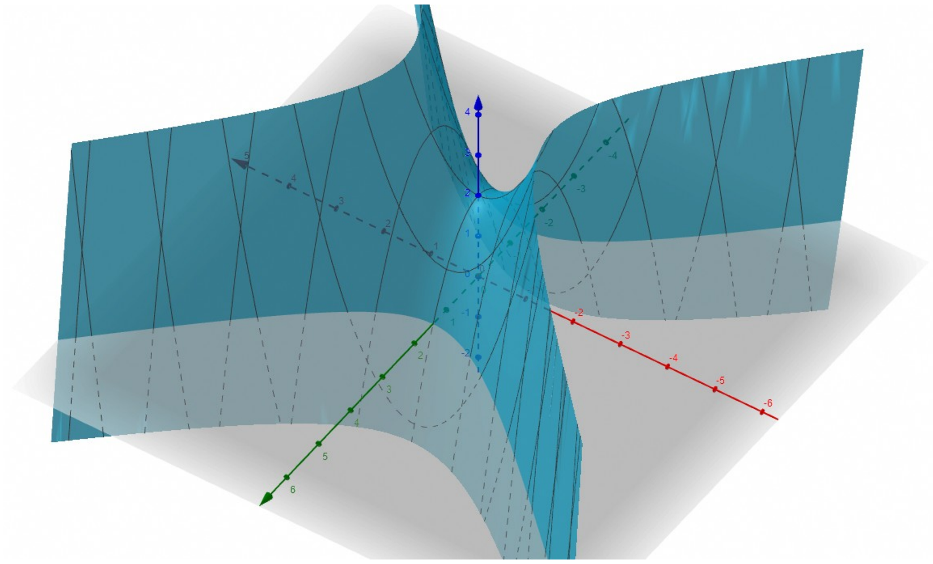
Visualization with geogebra shows that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is neither max nor min; it is a "monkey saddle".

Use https://www.geogebra.org/3d for generating the plots.

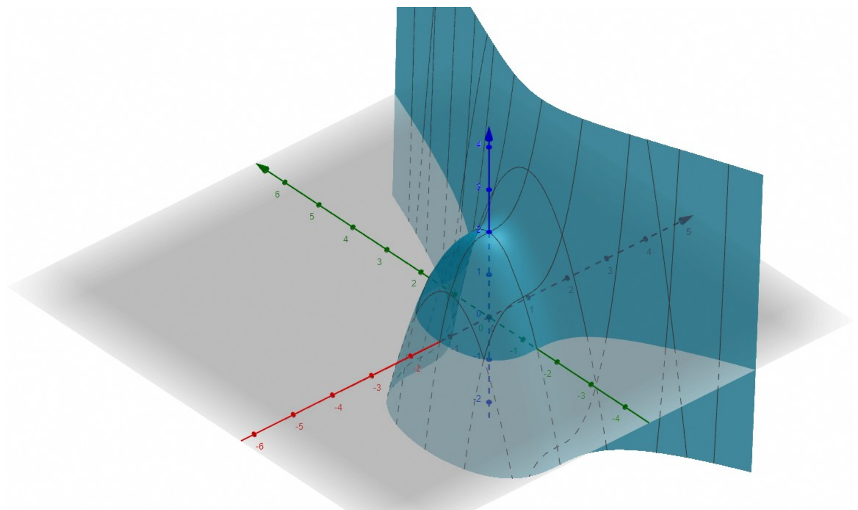
$f(x,y) = -x^2 - y^2 + 4$



$f(x,y) = x^2 - y^2 + 2$



$f(x,y) = x^3 - y^2 + 2$



$$f(x,y) = y^3 - 3x^2y + 2$$

("Monkey Saddle")