Elements of Calculus

Example Session Week 11

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Lecture notes from Spring 2025

Example Session for:

Week 11 A: Change of Variables, Differentials, Differential Operators

Week 11 B: Critical Points, Maxima and Minima

Exact and Inexact Differentials

Ex.: $dV = \underbrace{y}_{\frac{\partial V}{\partial x}} dx + \underbrace{x}_{\frac{\partial V}{\partial y}} dy$

Here, we see directly that $V(x,y) = xy + c$, so this differential is exact.

We also see this by checking $\frac{\partial f_1}{\partial y} = 1$ and $\frac{\partial f_2}{\partial x} = 1$.

Ex.: $dV = \underbrace{3y}_{\frac{\partial V}{\partial x}} dx + \underbrace{x}_{\frac{\partial V}{\partial y}} dy$

↳ From here, we need $V(x,y) = 3xy + g(y)$ (to satisfy $\frac{\partial V}{\partial x} = 3y$).

But then $\frac{\partial V}{\partial y} = 3x + \underbrace{\frac{\partial g}{\partial y}}_{\text{just fct. of } y}$, and this does not work, so this differential is inexact.

We also see this by checking $\frac{\partial f_1}{\partial y} = 3$, but $\frac{\partial f_2}{\partial x} = 1$.

Differential Operator Identities

Let us note some interesting identities in $\mathbb{R}^3$:

- curl grad $\varphi = \nabla \times (\nabla\varphi) = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \times \begin{pmatrix} \partial_{x_1}\varphi \\ \partial_{x_2}\varphi \\ \partial_{x_3}\varphi \end{pmatrix} = \begin{pmatrix} \partial_{x_2}\partial_{x_3}\varphi - \partial_{x_3}\partial_{x_2}\varphi \\ -\partial_{x_1}\partial_{x_3}\varphi + \partial_{x_3}\partial_{x_1}\varphi \\ \partial_{x_1}\partial_{x_2}\varphi - \partial_{x_2}\partial_{x_1}\varphi \end{pmatrix} \overset{\text{Clairaut / Schwarz}}{=} 0$

- div curl $f = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_2}f_3 - \partial_{x_3}f_2 \\ -\partial_{x_1}f_3 + \partial_{x_3}f_1 \\ \partial_{x_1}f_2 - \partial_{x_2}f_1 \end{pmatrix}$

$$= \partial_{x_1}(\partial_{x_2}f_3 - \partial_{x_3}f_2) + \partial_{x_2}(-\partial_{x_1}f_3 + \partial_{x_3}f_1) + \partial_{x_3}(\partial_{x_1}f_2 - \partial_{x_2}f_1) \underset{\text{Clairaut / Schwarz}}{=} 0$$

Optimization

Find critical points of $f(x,y) = x^2 + xy + y^2$ and discuss their properties.

Critical points: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \nabla f = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} (x^2 + xy + y^2) = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$

$\Rightarrow y = -2x$ and $y = -\frac{1}{2}x \Rightarrow x = 0 = y$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point.

The Hessian is $H_f(x,y) = \begin{pmatrix} \partial_x^2 f & \partial_x\partial_y f \\ \partial_y\partial_x f & \partial_y^2 f \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = H_f(0,0)$

What are the eigenvalues? $0 = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$

$\Rightarrow \lambda_{\pm} = 2 \pm \sqrt{4-3} = 2 \pm 1 \Rightarrow \lambda_+ = 3,\ \lambda_- = 1$

Both are positive, so the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a minimum.