Elements of Calculus

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Lecture notes from Spring 2025

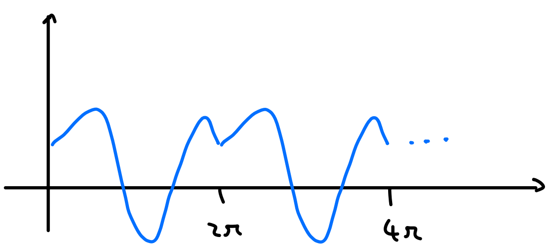
Session 27
(Week 14A)

8. Fourier Series

Topic for Week 14 A: Definition of Fourier Series and Examples

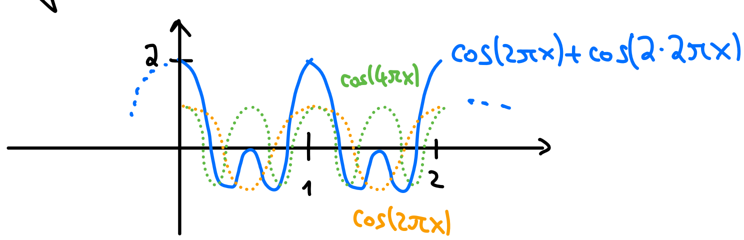
We consider 2π-periodic functions, i.e., $f(x+2\pi)=f(x)$

(L-periodic for any $0 \neq L \in \mathbb{R}$ works analogously).



The idea of Fourier series is to decompose periodic functions into "pure frequencies".

E.g., $f(x)=\cos(2\pi x)+\cos(2\cdot 2\pi x)$ is a sum of cosines with frequencies 1 and 2:



This is useful for many things:

- It works also for non-differentiable functions, so it can be a good alternative to Taylor series.
- It is very important for signal processing.
- It can be used to solve ODEs and PDEs.
- A generalization, the Fourier transform, also works for non-periodic functions.

Let us just consider one period, i.e., $f : [0, 2\pi] \to \mathbb{C}$, $f(0) = f(2\pi)$.

We assume f is Riemann-integrable on $[0, 2\pi]$.

Then the Fourier series of f is defined as $\mathcal{F}_f(x) := \sum_{k=-\infty}^{\infty} \underbrace{\hat{f}_k}_{=\text{Fourier coefficients}} \overbrace{e^{ikx}}^{=\cos kx + i \sin kx}$.

Note: $e_k(x) := e^{ikx}$ plays the role of a basis function.

Let us introduce the inner product $\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)}\, g(x)\, dx$ and norm $\|f\| = \sqrt{\langle f, f \rangle}$.

$$\text{Then } \langle e_j, e_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{e_j(x)}\, e_k(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-ijx} e^{ikx}\, dx = \frac{1}{2\pi} \begin{cases} \frac{1}{i(k-j)} e^{i(k-j)x} \Big|_0^{2\pi} = 0 & k \neq j \\ 2\pi & k = j \end{cases}$$

$$= \underbrace{\delta_{jk}}_{\text{Kronecker delta}} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$\Rightarrow$ $(e_k(x))_{k \in \mathbb{Z}}$ is an orthonormal set of vectors

Next, we compute: $\langle e_j, f \rangle = \langle e_j, \sum_{k=-\infty}^{\infty} \hat{f}_k e_k \rangle \underset{\substack{\text{Assuming we can}\\ \text{interchange summation}\\ \text{and integration!}}}{=} \sum_{k=-\infty}^{\infty} \hat{f}_k \underbrace{\langle e_j, e_k \rangle}_{\delta_{jk}} = \hat{f}_j$.

$\Rightarrow$ We can compute $\hat{f}_j$ if we can interchange summation and integration above (which we can do often, but not always).

So in that case: $\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x)\, dx$.

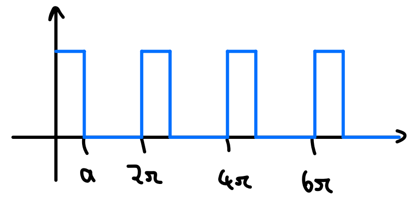
But we can define $\hat{f}_k$ for any Riemann integrable f.

So generally, we define the Fourier coefficients of f as $\hat{f}_k := \langle e_k, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x)\, dx$.

In the following, we want investigate whether $\sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$ always converges to $f(x)$, and if yes, in what sense?

First, we note that often $F_f(x)$ and $f(x)$ don't agree everywhere.

Example A: $f(x) = \begin{cases} 1 & \text{for } x \in [0,a) \\ 0 & \text{for } x \in [a, 2\pi) \end{cases}$ (square puls)



We find $\cdot\ \hat{f}_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = \frac{a}{2\pi}$

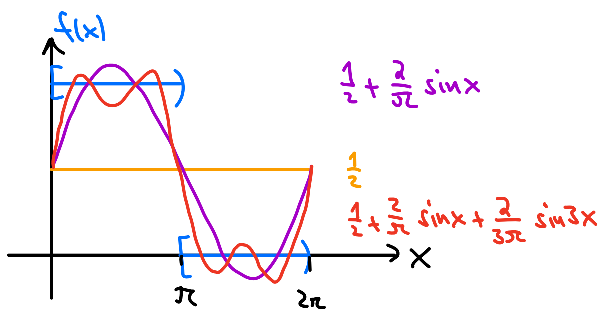
$\cdot$ For $k \neq 0$: $\hat{f}_k = \frac{1}{2\pi} \int_0^a e^{-ikx} f(x)\,dx = \frac{1}{2\pi} \int_0^a e^{-ikx}\,dx = \frac{1}{2\pi} \frac{1}{(-ik)} e^{-ikx} \Big|_0^a$

$$= \frac{i}{2\pi k}\left(e^{-ika} - 1\right)$$

E.g., for $a = \pi$, we have $\hat{f}_k = \frac{i}{2\pi k}\left(e^{-i\pi k} - 1\right) = \frac{i}{2\pi k}\left((-1)^k - 1\right) = \begin{cases} 0 & \text{for } k \text{ even} \\ \frac{-i}{\pi k} & \text{for } k \text{ odd} \end{cases}$

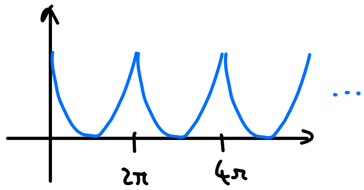
and $F_f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} = \underbrace{\frac{1}{2}}_{\hat{f}_0} + \sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{(-i)}{\pi k} e^{ikx} + \underbrace{\sum_{\substack{k=-\infty \\ k\,\text{odd}}}^{-1} \frac{(-i)}{\pi k} e^{ikx}}_{= -\sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{(-i)}{\pi k} e^{-ikx}} = \frac{1}{2} + \sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{(-i)}{\pi k} \underbrace{\left(e^{ikx} - e^{-ikx}\right)}_{\substack{i \sin kx - (-i \sin kx) \\ = 2i \sin kx}}$,

$\Rightarrow F_f(x) = \frac{1}{2} + \sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{2}{\pi k} \sin kx$.



Here, e.g., we see that $F_f(\pi) = \frac{1}{2} \neq f(\pi)$, so we do not have convergence at all points. But it looks like some type of convergence should hold.

Example B: $f(x) = (x-\pi)^2$ on $[0, 2\pi]$



A computation (see HW) shows $F_f(x) = \frac{\pi^2}{3} + \sum_{k \neq 0} \frac{2}{k^2} e^{ikx} = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos kx$,

which converges everywhere, i.e., $F_f(x) = f(x)$ $\forall x \in [0, 2\pi]$.

As a corollary we find $\sum_{k=1}^{\infty} \frac{4}{k^2} = f(0) - \frac{\pi^2}{3} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$, i.e., $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Question: What is the right kind of convergence for functions as in Example A?

We will investigate this in the next session.