Reconstruction of the Scattering Function of Overspread Radar Targets

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Abstract

This paper addresses the problem of stochastic radar target measurement. We develop an algorithm that allows for the reconstruction of the scattering function of a WSSUS radar target from the autocorrelation of the response of the target to a deterministic sounding signal. While conventional methods are applicable only when the scattering function is supported on a rectangle with area less than one, our method can handle area one support sets of arbitrary geometry.

Based on the suggested theoretical recovery procedure, we propose a scattering function estimator.

1 Introduction

In the classical delay-Doppler radar system, the echo $y(t)$ that is reflected from a target is expressed as a superposition of time-frequency shifts of the transmitted waveform $x(t)$. The contributions of each scatterer are represented by the spreading function $\eta(\tau, \gamma)$ of the target. If the target has randomly varying components, then the spreading function

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is a random field. It is often assumed that the radar environment satisfies the wide-sense
stationarity with uncorrelated scattering (WSSUS) assumption [1, 2]. The target is then
characterised by its scattering function $C(\tau, \gamma)$, comprising information about the pointwise
variance of $\eta(\tau, \gamma)$.

A problem in radar is to determine $C(\tau, \gamma)$ from the stochastic echo $y(t)$ [3, 4]. A
standard approach for an equivalent problem in wireless communications is to view the
scattering function as power spectral density of the 2-D stationary random process known
as the time-varying transfer function and then using an averaged periodogram estimator to
estimate $C(\tau, \gamma)$ [5, 6].

Ordinarily, the target is assumed to be underspread, that is, the support of $C(\tau, \gamma)$ is
required to be contained in a rectangle of area less than one. Alternatively, a target can
be bestowed with a (discrete) parametric model with fewer degrees of freedom [7, 8, 9].
Such models allow constructive analysis and quantitative results that exceed the general
limitations of arbitrary continuous-space systems, in particular, the restriction for the target
to be underspread. We show that this restriction is too strong for arbitrary continuous-space
targets, and can be replaced by the criterion that the area of the support of $C(\tau, \gamma)$ is less
than one.

Building on recent results in operator identification [10, 11, 12, 13, 14] and following
ideas from [15], we give a novel method to recover $C(\tau, \gamma)$ from the response of the stochastic

\[ \frac{\Omega}{2} = 2B \]

\[ \frac{-\Omega}{2} = -2B \]

Figure 1: Support of the scattering function. Given $BT = \frac{1}{8}$, and $J = 8$, since $\Omega\Theta = 16BT = 2$, this target is overspread.
operator to a deterministic sounding signal which does not require the bounding region of $C(\tau, \gamma)$ to be rectangular.

Our explicit reconstruction formula for a scattering function supported on a possibly overspread domain of the time-frequency plane using a specially constructed pulse train as the sounding signal allows us to control the effects of inevitable aliasing between responses to adjacent pulses. A procedure for the estimation of the scattering function is naturally derived from this reconstruction formula.

The paper is organised as follows. Section 2 gives an overview of classical delay-Doppler radar theory. In Section 3 we describe the target identification problem and prove the main result, Theorem 1. We give our conclusions in Section 5 and a few technical proofs in the appendix.

### 2 Overview of radar

Upon transmission of a continuous-time signal $x(t)$, the echo from a point target at a distance $d$ travelling at a constant speed $v$ relative to the receiver, has the form

$$y(t) = \eta_0 e^{2\pi i \gamma t} x(t - \tau) = \eta_0 M_\gamma T_\tau x(t),$$

where $\tau = 2d/c$, $c$ is the speed of light, $\gamma = v/c$ is the Doppler shift, and $\eta_0$ is the reflection coefficient that depends on the distance and the speed of the point target. Here, $T_\tau x(t) = x(t - \tau)$ is a time shift operator and $M_\gamma x(t) = e^{2\pi i \gamma t} x(t)$ is a frequency shift operator.

In an environment with multiple scatterers, the echo can be modelled as a continuous superposition of time-frequency shifts of the transmitted signal $x(t)$, that is,

$$y(t) = H x(t) = \int \int \eta(\tau, \gamma) M_\gamma T_\tau x(t) \, d\tau \, d\gamma,$$

where $\eta(\tau, \gamma)$ is the spreading function of the target. Equivalently to (1), from the

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1 For example, a Jakes/exponential scattering function is supported on a rectangular set $[0, \Theta] \times [-\Omega/2, \Omega/2]$ (where some error is already incurred by limiting it to a finite box), but it is predominantly zero between its branches, and hence its support can be taken to be a U-shaped set as on Figure 3b.
time-varying impulse response

\[ h(t, \tau) \triangleq \int \eta(\tau, \gamma) e^{2\pi i \gamma t} d\gamma \]  

we obtain the linear time-varying representation of the target

\[ y(t) = \int h(t, \tau) x(t - \tau) d\tau. \]

Certain characteristic features of a target or a radar scene, for example, sea clutter, are often modelled as random processes \([17, 2, 18, 19]\). In this model, \( h \) and \( \eta \) are assumed to be a pair of random processes related via a one-sided Fourier transform, see (2) below. The usual wide-sense stationarity with uncorrelated scattering (WSSUS) \([1, 2]\) assumption in radar applications is that \( h(t, \tau) \) is

\( a) \) zero-mean in both variables,

\( a) \) wide sense stationary (WSS) in \( t \), and

\( a) \) has uncorrelated scatterers \( \tau \),

which reads that the autocorrelation of \( h \) has the special form

\[ R_h(t_0, t_0 + t; \tau, \tau') \triangleq \mathbb{E} \{ h(t_0, \tau)^* h(t_0 + t, \tau') \} = P(t, \tau) \delta(\tau - \tau'), \]

where \( \mathbb{E} \) denotes expectation and \( P(t, \tau) \) is the autocorrelation function, or ACF, associated with the target. In this case, \( \eta(\tau, \gamma) \) satisfies the relation

\[ \mathbb{E} \{ \eta(\tau, \gamma)^* \eta(\tau', \gamma') \} = \delta(\tau - \tau') \delta(\gamma - \gamma') C(\tau, \gamma), \]

where \( C(\tau, \gamma) \) is called the scattering function of the target and satisfies

\[ C(\tau, \gamma) \triangleq \int P(t, \tau) e^{2\pi it\gamma} dt. \]
The computations above and in the remainder of this paper involve delta distributions, thus convergence and equality of integrals must be understood in a weak sense. We preserve current notation for clarity and relegate a rigorous functional analytic treatment of this matter to [20].

A classical identification problem in radar and communications is to estimate the scattering function of a given randomly varying target, or, equivalently, a linear time-variant random channel. The general approach is to transmit a signal \( x(t), \) once or multiple times, and to construct an estimator \( \hat{C}(\tau, \gamma) \) for the scattering function \( C(\tau, \gamma) \) using the received echoes \( y(t) \). For the reasons explained below, it is usually assumed that the scattering function \( C(\tau, \gamma) \) of a target has its support within a rectangle \([0, \Theta] \times [-\Omega/2, \Omega/2]\) on a time-frequency plane \((\tau, \gamma)\), that is, a radar scene has maximum time delay \( \Theta \) and maximum Doppler spread \( \Omega/2 \). The degree of dispersion of the echo is quantified by the spreading factor \( \Omega \Theta \). A target is called underspread if \( \Theta \leq 1 \) and overspread if \( \Theta > 1 \).

Contrary to a frequent assumption, we show that it suffices to require \( \text{supp} \ C(\tau, \gamma) \) to be a compact set of area \( \mu(\text{supp} \ C(\tau, \gamma)) \) less than one contained within some rectangle \([0, \Theta] \times [-\Omega/2, \Omega/2]\) without demanding \( \Theta \leq 1 \).

As we are going to use an ideal weighted impulse train \( x(t) = \sum_{k \in \mathbb{Z}} c_k \delta(t - kT) \) as a sounding signal, our procedure can be seen as a generalisation of Kailath sounding by a periodic delta train \( \sum_{k \in \mathbb{Z}} \delta(t - kT) \) to the case of non-rectangular support of the scattering function. In the works of Kailath [3], the impulses must be spaced far enough apart \( (T > \Theta) \) so that the responses to individual pulses do not overlap, and close enough \( (T < \frac{1}{\Theta}) \) to provide the sampling rate that is Nyquist for \( h(t, \tau) \) in \( t \) for every \( \tau \), hence the necessary condition \( \Theta \leq 1 \).

In case of a an overspread target, these requirements are impossible to achieve simultaneously, and the responses to individual impulses sent at the Nyquist rate will necessarily overlap. We prove that this aliasing can be controlled and provide a method to revert its effects in the case when the total area of the support set \( \text{supp} \ C(\tau, \gamma) \) is less than one.

We acknowledge that in practice, infinite duration and large crest factor of the input

\footnote{Obtaining measurements in this way is referred to as channel sounding in communications.}
signal are undesirable. These drawbacks can be mitigated by modifying the sounding signal. On particular, time-gating the impulse train controls the duration, and band limiting it, or convolving with a smooth function, controls the crest factor. Such modifications result in the method that is capable of resolving/identifying an operator action on a portion of the time-frequency plane [21, 22].

3 Scattering function identification

Let \( \eta(\tau, \gamma) \) be the spreading function of a radar target such that \( \text{supp} C(\tau, \gamma) \subseteq [0, \Theta] \times [-\Omega/2, \Omega/2] \), with \( \Omega \Theta \) possibly larger than 1. Let the set \( \text{supp} C(\tau, \gamma) \) be covered by \( J \) translations, of a prototype rectangle \( R = [0, T] \times [0, B] \), that is,

\[
\text{supp} C(\tau, \gamma) \subseteq \bigcup_{j=1}^{J} [a_j T, (a_j + 1)T) \times [b_j B, (b_j + 1)B)
\]

where \( a_j, b_j \in \mathbb{Z} \) index the cover, \( T = \Theta/\Omega, B = \Omega/J\Theta \). This implies \( JBT = 1 \). Note that other bounding boxes, for example, a first quadrant box \( [0, \Theta] \times [0, \Omega] \), can be accommodated by translation. Consequently, such covering corresponds to a patch decomposition

\[
C(\tau, \gamma) = \sum_{j=1}^{J} C_j(\tau - a_j T, \gamma - b_j B),
\]

where \( C_j(\tau, \gamma) \equiv \chi_R(\tau, \gamma) C(t + a_j T, \gamma + b_j B) \).

The case \( JBT < 1 \) can be padded with empty boxes, if necessary. Examples of such covering can be found on Figure 3. The theory of Jordan domain shows that for any compact set \( M \) of area less than one such covering \( \{(a_j, b_j)\}_{j=1}^{J} \) can always be found. We define patches of \( \eta \) to be 2D random processes

\[
\eta_j(\tau, \gamma) \equiv \chi_R(\tau, \gamma) \eta(\tau + a_j T, \gamma + b_j B),
\]

and their Fourier transforms

\[
h_j(t, \tau) \equiv \int \eta_j(\tau, \gamma) e^{2\pi i \tau t} \, d\gamma.
\]

Consequently, the autocorrela-
tion function $P(t, \tau)$ admits an expansion

$$P(t, \tau) = \sum_{j=1}^{J} e^{-2\pi i \tau B_j} P_j(t, \tau - a_j T),$$

(3)

where $P_j(t, \tau) \triangleq \int e^{-2\pi i \gamma t} C_j(\tau, \gamma) d\gamma$.

Using Lemma 2 and Lemma 3 from the Appendix, we are now ready to prove the main result of this paper.

**Theorem 1.** Let $C(\tau, \gamma)$ be the scattering function of a radar target. Let the support of $C(\tau, \gamma)$ have area $M = \mu(\text{supp} C(\tau, \gamma)) < 1$. There exists $T > 0$, a natural number $J$ and a fixed $J$-periodic sequence $\{c_k\}_{k \in \mathbb{Z}}$ such that we can identify the scattering function $C(\tau, \gamma)$ from the received echo $y(t) = Hx(t)$ where $x(t) = \sum_{k \in \mathbb{Z}} c_k \delta(t - kT)$ is the sounding signal.

**Proof.** Let $\{c_k\}_{k \in \mathbb{Z}}$ be a $J$-periodic complex sequence. Transmit $x(t) = \sum_{k \in \mathbb{Z}} c_k \delta(t - kT)$ and observe echo $y(t) = Hx(t)$. Also, define

$$y_n(t) \triangleq (Hx)(t + nT) = \sum_{k \in \mathbb{Z}} c_k \ h(t + nT, t + (n - k)T),$$

(4)

for $0 \leq t < T$ and $n \in \mathbb{Z}$. Clearly, if $h(t, \tau)$ is a stochastic process, so are the $y_n(t)$. For $0 \leq t, \tau < T$ and $n = mJ + r \in \mathbb{Z}$, by Lemma 3 we have

$$\mathbb{E} \{y_{mJ+r}(t)^* y(\tau)\} = \delta(t - \tau) \Pi_{mJ+r}(\tau),$$

(5)

with $\Pi_{mJ+r}$ defined in [13].

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3In [20, 23], the class of stochastic operators satisfying $\text{supp} C \subseteq M$ is referred to as stochastic operator Paley-Wiener space and is denoted $\text{StOPW}(M)$. 
Recall $JBT = 1$ and denote for $r = 1, \ldots, J$ and compute

$$S_r(\tau, \gamma) \triangleq \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ+r)} \Pi_{mJ+r}(\tau)$$

$$= \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ+r)} \Pi_{mJ+r}(\tau)$$

$$= \sum_{j=1}^{J} e^{-2\pi i r b_j / J} \sum_{k \in \mathbb{Z}} c_k^* c_k$$

$$\times \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ+r)} P_j((mJ + r)T, \tau - (k + a_j)T)$$

$$= \sum_{j=1}^{J} e^{-2\pi i r b_j / J} \sum_{k \in \mathbb{Z}} c_k^* c_k C_j(\tau - (k + a_j)T, \gamma),$$

where we used Lemma 2.

But, $C_j(\tau, \gamma) \neq 0$ only if $0 \leq \tau < T$. Then, $C_j(\tau - kT - a_jT, \gamma) \neq 0$ if and only if $\tau - T < (k + a_j)T \leq \tau$. Since $0 \leq \tau < T$, we must have that $C_j(\tau - (k + a_j)T, \gamma) \neq 0$ only if $k + a_j = 0$. Therefore, by (6),

$$S_r(\tau, \gamma) = \sum_{j=1}^{J} e^{-2\pi i r b_j / J} c_{r-a_j}^* c_{-a_j} C_j(\tau, \gamma).$$

(7)

Let $G = \{g_{k,l}\}_{k,l=0}^{J-1}$ be the Weyl-Heisenberg-Gabor frame for $\mathbb{C}^J$ with window $[c_r]_{r=1}^J$, that is,

$$g_{k,l}(r) = e^{-2\pi i r l / J} c_{r-k}, \quad r = 1, \ldots, J,$$

and by $U$ the $J \times J$ submatrix of $G$ whose $j$-th column is $c_{a_j, b_j}$, that is,

$$U = [c_{a_1, b_1} | c_{a_2, b_2} | \cdots | c_{a_J, b_J}]$$

The coefficient sequence $c \triangleq \{c_r\}_{r=1}^J$ (and hence, the entire $J$-periodic infinite sequence $\{c_k\}_{k \in \mathbb{Z}}$ can be chosen so that any $J$ element subset out of $J^2$ of the so called Gabor frame vectors $G$ is linearly independent [24] [25], a condition of the frame known as the Haar property. Moreover, the selection procedure allows us to choose $c$ to be unimodular. In fact,
choosing the entries of \( c \) randomly from a uniform distribution on a unit circle guarantees the Haar property will hold almost surely \cite{21}. In particular, this means that the matrix of our interest \( U \), determined by the geometry of the support set of \( C(\tau, \gamma) \), is invertible. Denoting by \( C \) the column vector of patches of the scattering function

\[
C(\tau, \gamma) = [C_j(\tau, \gamma)]_{j=1}^J.
\]

defined on \((\tau, \gamma) \in [0, T) \times [0, B)\), \cite{7} can be expressed in matrix form as

\[
[S_r(\tau, \gamma)]^{J}_{r=1} = U^*[c_{-a_j}C_j(\tau, \gamma)]^{J}_{j=1},
\]

so that we can rewrite the above equation as

\[
[S_r(\tau, \gamma)]^{J}_{r=1} = U^*D_cC(\tau, \gamma).
\]

with \( D_c \in \mathbb{C}^{J \times J} \) a non-singular diagonal matrix with elements of \( c \) on the diagonal, that is, \( D_{ij} = c_{-a_j}\delta_{ij} \). Let \( V = [V_{jr}]_{r,j=1}^J \) be the inverse of \( U^*D_c \). We can now recover \( C_j(\tau, \gamma) \) pointwise from the autocorrelation of the output signal with an explicit reconstruction formula derived from \cite[4]{4}, \cite[5]{5} and \cite[6]{6}. For any \( j \in [1, J] \) and \( \tau \in [0, T), \gamma \in [0, B) \) we have

\[
C_j(\tau, \gamma) = \sum_{r=1}^J V_{jr} S_r(\tau, \gamma)
\]

\[
= \sum_{r=1}^J V_{jr} \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ+r)} \Pi_{m,J+r}(\tau)
\]

where we recall \( \Pi_m(\tau)\delta(\tau-\xi) = \mathbb{E} \{ y(t + nT)^* y(\xi) \} \). This completes the proof of Theorem 1.

We can simplify the expression \cite{9} by observing from the definition \cite{6} that

\[
S_r(\tau, \gamma) \delta(\tau - t)
\]
\[
= \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ + r)} \Pi_{mJ + r}(\tau) \delta(\tau - t)
= \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ + r)} \mathbb{E}\{y(\tau + (mJ + r)T)^* y(t)\}
= \frac{1}{B} \mathbb{E}\{ e^{2\pi i \gamma T} Z y(\tau + rT, \gamma)^* y(t)\},
\]

where Zak transform \(Z : L^2(\mathbb{R}) \to L^2([0, JT] \times [0, B])\) is defined by
\[
Z y(\tau, \gamma) \triangleq \sum_{m \in \mathbb{Z}} y(\tau - mJT) e^{2\pi imJT \gamma}.
\]

We form a column vector \(Y(\tau, \gamma, t) = [Y(\tau, \gamma, t)]^j_{\ell=1} \triangleq \mathbb{E}\{ e^{2\pi i \gamma T} Z y(\tau + rT, \gamma)^* y(t)\}\) of correlates of the received signal with its Zak transform, plug it into (8) and invert \(U^* D_c\) to get the matrix form of (9)
\[
C(\tau, \gamma) \delta(\tau - t) = V Y(\tau, \gamma, t).
\]

### 3.1 Estimation procedure

Although \(\text{Theorem 1}\) enables us to reconstruct the scattering function perfectly from the stochastic process \(y(t)\), in practice we do not know the values of \(\mathbb{E}\{y(\tau)^* y(t)\}\).

To address this problem, we assume the availability of a stochastic ensemble of identical targets and sound each with the same input signal \(x(t) = \sum_{k \in \mathbb{Z}} c_k \delta(t - kT)\) to obtain \(L\) independent identically distributed samples of target echoes \(y^{(\ell)}(t), \quad \ell = 1, \ldots, L\).

We consider as data for our estimator the functions \(\Pi_n(\tau)\) defined in (5) for \(\tau, t \in [0, T]\) as follows.
\[
\hat{\Pi}_{mJ + r}(\tau, t) = \frac{1}{L} \sum_{\ell=1}^{L} y^{(\ell)}(\tau - (mJ + r)T)^* y^{(\ell)}(t).
\]
Clearly, plugging the above into (9) induces an estimator \(\hat{C}(\tau, t, \gamma)\) of the scattering function \(C(\tau, \gamma)\), with components \(\hat{C}_j(\tau, \gamma)\) found from
\[
\hat{C}_j(\tau, t, \gamma) = \sum_{\ell=1}^{j} V^*_{j\ell} \frac{1}{B} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ + r)} \hat{\Pi}_{mJ + r}(\tau, t),
\]
where the dependence on \( t \) is degenerate (only via a factor \( \delta(t - \tau) \)). The constructed estimator is unbiased in a sense that

\[
E \hat{C}_j(\tau, t, \gamma) = \sum_{r=1}^{J} V_{jr} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ+r)} \times \frac{1}{L} \sum_{\ell=1}^{L} E \left\{ y^{(\ell)}(\tau - (mJ + r)T)^* y^{(\ell)}(t) \right\} \\
= \sum_{r=1}^{J} V_{jr} \sum_{m \in \mathbb{Z}} e^{2\pi i \gamma T(mJ+r)} \Pi_{mJ+r}(\tau) \delta(\tau - t) \\
= C_j(\tau, \gamma) \delta(\tau - t).
\]

We do not currently compute the variance of this estimator.

4 Simulation

We model all scatterers as independent random variables Gaussian with mean zero and variance determined by the scattering function \( C(\tau, \gamma) \) given on a mesh with step sizes \( M, N \) in time and frequency directions, respectively. That is, we consider a discrete target \( H \) with a random spreading function supported within a bounding rectangle on the time-frequency plane \([0, \Theta] \times [0, \Omega]\)

\[
\eta[k, \ell] \triangleq \eta \left( \frac{k}{\Theta}, \frac{\ell}{\Omega} \right), \quad k = 1, \ldots, \Theta M, \quad \ell = 1, \ldots, \Omega N.
\]

In our examples we take \( \Theta = 1 \text{ sec} \) and \( \Omega = 3 \text{ Hz} \), so that \( \Theta \Omega > 1 \), corresponding to a target that would classically be considered overspread. We run the algorithm with \( C(\tau, \gamma) \) being one of the two choices, i) an artificial example of a sum of a characteristic function of a certain rectangle and a certain Gaussian

\[
C_1(\tau, \gamma) = \alpha \chi_R(\tau, \gamma) + \beta \exp \left( -\frac{(\tau - a)^2}{2\sigma_{\tau}^2} - \frac{(\gamma - b)^2}{2\sigma_{\gamma}^2} \right)
\]
and ii) Jakes-exponential model given by $C_2(\tau, \nu) = \frac{1}{\rho^2} \int Q(\nu) P(\tau) \, d\tau$ with delay power profile $P(\tau)$ and Doppler power profile $Q(\nu)$ given by

$$P(\tau) = \frac{\rho^2}{\tau_0} e^{-\tau/\tau_0} \chi_{[0, \Theta]}(\tau),$$

$$Q(\nu) = \frac{\rho^2}{\pi \sqrt{\Omega^2 - (\nu - \Omega/2)^2}} \chi_{[0, \Omega]}(\nu),$$

where the trivial shift of the spectrum to the first quadrant is done purely for cosmetic reasons. We illustrate both choices on Figure 2.

![Figure 2: Scattering functions](image)

Observe that in both cases, the bounding rectangle $[0, \Theta] \times [0, \Omega]$ has large area (greater than one), which cannot be reduced by translating or adjusting the margins for tighter fit, but the numerically significant support of the scattering function can be contained in a union of rectangles of total area smaller than one, shown with black outlines.

For each scattering function, we simulate multiple soundings of a target $H$ with a (time-gated) weighted discrete delta train. For each sounding, we compute the discrete echo $y$ on interval $[-T, T]$ with a mesh size $1/M$ and averages $\bar{\Pi}_{m,J+r}(\tau, t)$ as in (10). The estimator for the scattering function is then recovered according to (11).

The normalised mean square error of the estimator for up to 1000 soundings is shown on Figure 4.
Figure 3: Scattering functions before and after reconstruction

Figure 4: Mean-square error
5 Conclusion

In this paper, we presented a method to recover the scattering function of a target using the second order statistics of the returned echoes from sounding by a custom weighted delta train, provided that \( \text{supp} \ C(\tau, \gamma) \) is contained in a union of rectangles of total area 1. This includes targets commonly considered overspread under the classic criterion that the area of the bounding box is less than 1. In the abstract setting of a continuous WSSUS target sounded by an infinite length pulse train, given complete second-order statistics of the received signal, we guarantee exact recovery of the scattering function. We suggest that the proposed technique to reshuffle the patches of the support of the scattering function can and should be seen as a generalisation of the Zak transform in the efficient implementation of the averaged periodogram estimator procedure offered in [6]. We note that although our results are obtained in a continuous-time setting, analogous results are true under the assumption of discrete time, with Fourier and Zak transforms taken over finite groups instead.

For completeness, we give an explicit recipe for a simple unbiased estimator. In the consequent paper [22], we offer a slightly different estimator, where due to its streamlined structure, we can provide a bound on the variance. Besides statistical properties of such estimators, it would be interesting to explore the stability of the technique under time-gating of the input signal. We currently do not pursue this direction, since the purpose of this paper is to establish new possibilities for scattering function reconstruction and to demonstrate the usefulness of the weighted pulse train technique in the field of radar and sonar acquisition.

A Appendix

Lemma 2. With \( \eta_j, h_j, P_j \) and \( C_j \) defined as above, we have

\[
C_j(\tau, \gamma) = \frac{1}{B} \sum_{n \in \mathbb{Z}} e^{2\pi i \gamma (t + n/B)} P_j(t + \frac{n}{B}, \tau),
\]

where \( t \) is arbitrary.
Proof. For every $\tau \in [0, T)$, $\eta_j(\tau, \gamma)$ is supported within $[0, B)$. Therefore, for any fixed parameter $t$ we have the orthonormal Fourier expansion

$$\eta_j(\tau, \gamma) = \frac{1}{B} \sum_{n \in \mathbb{Z}} \left( \int_{0}^{B} \eta_j(\tau, s) e^{2\pi i s (\xi + \frac{n}{B})} ds \right) e^{-2\pi i \gamma (t + \frac{n}{B})}$$

$$= \frac{1}{B} \sum_{n \in \mathbb{Z}} h_j \left( t + \frac{n}{B}, \tau \right) e^{-2\pi i \gamma (t + \frac{n}{B})}.$$

By using the patch expansion of $P(t, \tau)$ [3],

$$E \{ \eta_j(\tau, \gamma)^* \eta_j(\tau', \gamma') \}$$

$$= \frac{1}{B^2} \sum_{n, m \in \mathbb{Z}} e^{2\pi i (\gamma + \gamma') (n/m - m/m)} E \{ h_j(t, \tau)^* h_j(t', \tau') \}$$

$$= \frac{1}{B^2} \sum_{n, m \in \mathbb{Z}} e^{2\pi i (\gamma + \gamma') (n/m - m/m)} P_j \left( t + \frac{n-m}{B}, t \right) \delta(\tau - \tau')$$

$$= \frac{1}{B^2} \sum_{n, m \in \mathbb{Z}} e^{2\pi i (\gamma + \gamma') (n/m - m/m)} P_j \left( t + \frac{n}{B}, t \right) \delta(\tau - \tau')$$

$$= \frac{1}{B} \sum_{n \in \mathbb{Z}} e^{2\pi i (\gamma + \gamma') (n/m)} P_j \left( t + \frac{n}{B}, \tau \right) \delta(\tau - \tau') \delta(\gamma - \gamma').$$

Since $E \{ \eta_j(\tau, \gamma)^* \eta_j(\tau', \gamma') \} = \delta(\tau - \tau') \delta(\gamma - \gamma') C_j(\tau, \gamma)$, the result follows. \hfill \qed

Lemma 3. In the notations defined above,

$$E \{ y_{mJ+r}(t)^* y(\tau) \} = \delta(t-\tau) \sum_{j=1}^{J} e^{-2\pi i r b_j / J}$$

$$\times \sum_{k \in \mathbb{Z}} c_{k+r} c_k P_j \left( \frac{m}{B} + rT, \tau - (k + a_j)T \right).$$

Proof. For $0 \leq t, \tau < T$ and $n = mJ + r \in \mathbb{Z}$, we have

$$E \{ y_n(t)^* y(\tau) \}$$

$$= \sum_{k, l \in \mathbb{Z}} c_k^* c_l E \{ h(t + nT, t + (n-k)T)^* h(\tau, \tau - lT) \}$$

$$= \sum_{k, l \in \mathbb{Z}} c_k^* c_l P(nT + t - \tau, t + (n-k)T)$$

$$= \sum_{k, l \in \mathbb{Z}} c_k^* c_l P_{mJ+r}(t, \tau).$$
\( \times \delta(t - \tau - (n - k + \ell)T). \)

But for 0 \( t \leq \tau < T, \)

\[
\delta(t - \tau - (n - k + \ell)T) = \delta(t - \tau) \delta(n - k + \ell).
\]

Thus

\[
\mathbb{E} \{ y_n(t)^* y(\tau) \} \\
= \sum_{k \in \mathbb{Z}} c_k^* c_{k-n} P(nT, t + (n-k)T) \delta(t - \tau) \\
= \sum_{k \in \mathbb{Z}} c_{k+n}^* c_k P(nT, t - kT) \delta(t - \tau) \\
= \Pi_n(\tau) \delta(t - \tau),
\]

where we defined

\[
\Pi_n(\tau) \triangleq \sum_{k \in \mathbb{Z}} c_{k+n}^* c_k P(nT, \tau - kT). \quad (13)
\]

Using (9) and (12), we obtain

\[
\Pi_{r+mJ}(\tau) \\
= \sum_{k \in \mathbb{Z}} c_{k+r+mJ}^* c_k P(m/B + rT, \tau - kT) \\
= \sum_{k \in \mathbb{Z}} c_{k+r}^* c_k \sum_{j=1}^J e^{-2\pi i(m/B+rT)b_j B} P_j(m/B + rT, \tau - (k + a_j)T) \\
= \sum_{j=1}^J e^{-2\pi i r b_j / J} \sum_{k \in \mathbb{Z}} c_{k+r}^* c_k P_j(m/B + rT, \tau - (k + a_j)T). \quad \square
\]

References


