1. Topological vector spaces

Let $V$ be a vector space. An infinite family of elements from $V$ is called \textit{linearly independent} if any finite subfamily of it is linearly independent. A maximal linearly independent family in $V$ is called a \textit{Hamel basis} of $V$. The cardinality of a Hamel basis is called the \textit{algebraic dimension} of $V$.

Let $K$ denote the field of scalars (usually $K = \mathbb{R}$ or $\mathbb{C}$). A subset $E \subseteq V$ is called \textit{balanced} if $\forall \alpha \in K$ such that $|\alpha| \leq 1$, we have $\alpha E \subseteq E$. A subset $E$ is called \textit{absorbing} if $\bigcup_{\lambda \in K} \lambda E = V$.

A vector space $V$ is called a \textit{topological vector space} if there is a topology in $V$ such that the linear operations (addition and multiplication by scalars) are continuous.

A map $p : V \to [0, \infty]$ is called a \textit{semi-norm} if $p(\lambda x) = \lambda p(x)$ (positive homogeneity) and $p(x + y) \leq p(x) + p(y)$ (convexity). A semi-norm is called a \textit{norm} if it is finite everywhere and $p(x) = 0$ implies $x = 0$. If no confusion can arise, a norm is denoted by $\| \cdot \|$.

For a semi-norm $p$, define the \textit{unit ball} $B_p = \{ x \in V \mid p(x) \leq 1 \}$. For each absorbing subset $B \subseteq V$, define the \textit{Minkowski functional} of $B$ as $p_B(x) = \inf\{ \lambda > 0 \mid x \in \lambda B \}$ (if there is no such $\lambda$ then $p_B(x) = \infty$).

A topological vector space $V$ is called \textit{locally convex} if it has a basis of convex neighborhoods of 0. If the topology of $V$ is defined by a family of semi-norms, then the space $V$ is said to be \textit{polynormed}. If this family of semi-norms is countable, then $V$ is said to be \textit{countably normed}. A topological vector space is called a \textit{metric vector space} if there is a translation invariant metric on $V$ (translation invariance of a metric $d$ means that $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in V$). A space equipped with a norm is called a \textit{normed space}. A complete normed space is called a \textit{Banach space}.

Problems and theorems.

1.1. Prove that algebraic dimension of a vector space does not depend on the choice of a Hamel basis.

1.2. Vector spaces are isomorphic iff they have the same algebraic dimension.

1.3. A subset $B \subseteq V$ is convex, balanced and absorbing iff $p_B$ is a semi-norm and $B = B_p$.

1.4. A subset $B$ is convex, balanced, absorbing and does not contain any line iff $p_B$ is a norm and $B = B_p$.

1.5. In a topological vector space, for any neighborhood $U$ of 0, there exists a neighborhood $V$ such that $V + V \subseteq U$.

1.6. In a topological vector space, for any neighborhood $U$ of 0, there exists a neighborhood $V$ such that $V = -V$ and $V + V \subseteq U$. 1
1.7. An arbitrary neighborhood of 0 in a locally convex space contains an open convex balanced neighborhood. *Hint:* use the continuity of scalar multiplication.

1.8. Any locally convex space is polynormed. Any polynormed space is locally convex.

1.9. Any countably normed space is metric. *Hint:* define

\[ d(x, y) = \sum 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}. \]

1.10. Suppose that, in a topological vector space \( V \), all points are closed. Then any compact set \( K \subset V \) can be separated from any closed set \( C \subset V \) disjoint with \( K \). In other terms, there are open sets \( U \supset K \) and \( V \supset C \) such that \( U \cap V = \emptyset \).

1.11. If, in a topological vector space, all points are closed, then this space is Hausdorff.

1.12. Let \( V \) be a non-Hausdorff space. There exists a unique subspace \( W \) such that any neighborhood of any point in \( W \) contains \( W \), and \( V/W \) is Hausdorff. If \( V \) is polynormed with the family \( \{p_\alpha\} \) of semi-norms, then \( W = \{all p_\alpha = 0\} \). In the sequel, we assume all topological vector spaces to be Hausdorff.

1.13. A linear map between topological vector spaces is continuous iff it is continuous at 0.

1.14. Consider a linear functional \( l : V \to \mathbb{K} \) on a topological vector space. The following statements are equivalent:
- \( l \) is continuous,
- \( \ker(l) \) is closed,
- \( \ker(l) \) is not everywhere dense.

1.15. **The Hahn-Banach theorem.** Let \( p \) be a semi-norm on \( V \) and \( W \subseteq V \) a subspace. Suppose \( f \) is a functional on \( W \) such that \( |f| \leq p \) on \( W \). Then \( f \) extends to a functional on \( V \) such that \( |f| \leq p \) on \( V \). *Hint:* use Zorn’s lemma.

1.16. In a polynormed space, continuous linear functionals separate points. In other words, for any pair of points \( x, y \in V \), there exists a continuous linear functional \( l : V \to \mathbb{K} \) such that \( l(x) < 0 < l(y) \).

1.17. Let \( V \) be a real vector space. Consider a convex and absorbing set \( B \subset V \) and a point \( y \not\in B \). There is a linear functional \( l \) on \( V \) such that \( |l|_B \leq 1 \) and \( l(y) > 1 \). (In other terms, \( B \) can be separated from \( y \) by a hyperplane). *Hint:* one can assume WLOG that \( B \) is balanced, otherwise replace \( B \) with \( B \cap (-B) \).

1.18. **Geometric form of the Hahn–Banach theorem.** Let \( V \) be a real vector space. The kernel of a subset \( X \subseteq V \) consists of all points \( x \in X \) such that

\[ (\forall y \in V)(\exists \varepsilon > 0)(\forall t \in \mathbb{R} : |t| < \varepsilon) \ x + ty \in X. \]

(equivalently, the kernel of \( X \) is the set of all \( x \in X \) such that \( X - x \) is absorbing). Let \( X \subseteq V \) be a convex subset of \( V \) with a nonempty kernel and \( Y \subseteq V \) - a convex subset such that \( X \cap Y = \emptyset \). Then there exists a hyperplane separating \( X \) and \( Y \). *Hint:* consider the Minkowski functional of \( (X - x) - (Y - y) \) where \( x \) lies in the kernel of \( X \) and \( y \in Y \).

1.19. **The Banach-Steinhaus theorem.** Let \( V \) be a complete metric vector space and \( W \) a normed space. Consider a family of operators \( A_\gamma : V \to W \). If for any \( x \in V \) the family \( \{A_\gamma(x)\} \) is bounded, then the family \( \{A_\gamma\} \) is uniformly bounded on the unit ball. *Hint:* consider the set

\[ F_k = \{ x \in V | \|A_\gamma(x)\| \leq k \ \forall \gamma \}. \]

We have \( V = \bigcup F_k \). By the Baire theorem, at least one \( F_k \) has nonempty interior. Therefore, \( A_\gamma \) is uniformly bounded on some ball.
1.20. In notation of the previous problem, \( \{A_n\} \) is equicontinuous.

1.21. **Banach’s open mapping theorem.** Let \( V \) and \( W \) be Banach spaces. Then each one-to-one continuous operator \( A : V \to W \) is open, i.e., the inverse operator \( A^{-1} \) is continuous.

**Plan of the proof.** Let \( B \subset V \) be an open ball of radius \( \varepsilon \) centered in 0. Define \( X_n = A(nB) \). Prove that at least one of \( X_n \) contains a ball (use the Baire property). Prove that \( \bar{A}(B) \) contains a ball centered in 0. Finally, prove that \( A(B) \) contains a ball centered in 0.

1.22. Suppose a space \( V \) is complete with respect to both norms \( p_1 \) and \( p_2 \) and \( p_1 \leq Cp_2 \). Then \( p_2 \leq C'p_1 \), i.e., the norms \( p_1 \) and \( p_2 \) are equivalent. **Hint:** consider the identity map from \((V, p_1)\) to \((V, p_2)\).

1.23. Let \( Z \) be a Banach space, \( X \) and \( Y \) closed subspaces s.t. \( Z = X + Y \), \( X \cap Y = 0 \). Then the projection to \( X \) is continuous. **Hint:** consider the norm \( z \mapsto ||\pi_X(z)|| + ||\pi_Y(z)|| \), where \( \pi_X \) and \( \pi_Y \) are the projections to \( X \) and \( Y \), respectively.

1.24. **F. Riesz’ theorem.** Let \( X \) be a normed space such that there is a compact set \( B \), whose interior contains the origin. Then \( X \) is finite-dimensional. (In other terms, any locally compact normed space is finite-dimensional). **Hint:** there is a finite set \( A \) such that \( B \subseteq A + \frac{1}{2}B \). Let \( Y \) be the vector subspace spanned by \( A \). Show that \( B \subseteq Y + 2^{-n}B \) for any natural \( n \). Deduce that \( B \subseteq Y \).

2. **Dual spaces**

For a topological vector space \( V \), let \( V^* \) denote the space of all continuous linear functions on \( V \). The space \( V^* \) is called the dual space of \( V \). If \( V \) is a normed space, then there is a norm in \( V^* \):

\[
||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||}.
\]

If \((V, \{p_\alpha\})\) is a polynormed space, then \( V^* \) is also polynormed with respect to the family of semi-norms

\[
p^*_\alpha = \sup_{x \neq 0} \frac{|p_\alpha(x)|}{||x||}.
\]

The weak topology in a topological vector space \( V \) is defined by a family of semi-norms \( p_f(x) = |f(x)| \) where \( f \) runs over \( V^* \). The convergence with respect to the weak topology is called the weak convergence (notation: \( x_n \xrightarrow{w} x \)). We have \( x_n \xrightarrow{w} x \) iff \( f(x_n) \to f(x) \) for any \( f \in V^* \). The \(*\)-weak topology in \( V^* \) is defined by a family of semi-norms \( p_x(f) = |f(x)| \) where \( x \) runs over \( V \). If \( V \) is a normed space, then the topology in \( V^* \) defined by the norm in \( V^* \) is called the strong topology. By default, the topology in \( V^* \) is assumed to be strong.

**Problems and theorems.**

2.1. For any normed space \( V \) (not necessary complete) the dual space \( V^* \) is complete.

2.2. Let \( V \) be a normed space. Then for any \( x \in V \) there exists \( f \in V^* \) such that \( |f(x)| = ||f|| \cdot ||x|| \). **Hint:** use the Hahn-Banach theorem.

2.3. For a normed space \( V \), the embedding \( V \to V^{**} \) is isometric.
2.4. Let $V$ and $W$ be normed spaces and $A : V \to W$ a continuous linear operator. Define

$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}.$$ 

Prove that $||A^*|| = ||A||$, where $A^* : W^* \to V^*$ is the dual operator defined by the formula $(A^* f)(x) = f(Ax)$.

2.5. **The Banach-Alaoglu theorem.** Let $V$ be a separable normed space. Then the closed unit ball in the dual space $V^*$ is compact in the $*$-weak topology. Hint: Let $X$ be a countable dense set in $V$. Consider a sequence of functionals $l_n \in V^*$, $||l_n|| \leq 1$. For each $x$, the sequence $l_n(x)$ contains a convergent subsequence. Use the diagonal argument to find a subsequence $l_n(x)$ such that $l_n(x)$ converges to some number $l(x)$ for each $x \in X$. Prove that $l$ extends to a well-defined linear functional on $V$ with $||l|| \leq 1$.

### 3. Hilbert spaces

A Hilbert space is a vector space $H$ with a positively definite hermitian metric $\langle \cdot, \cdot \rangle$ such that $H$ is complete with respect to the norm $||x|| = \sqrt{\langle x, x \rangle}$.

**Problems and theorems.**

3.1. **The parallelogram law.** A normed space $V$ is a Hilbert space iff

$$(\forall x, y \in V) \quad ||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$ 

3.2. Let $K$ be a nonempty convex closed subset in a Hilbert space $H$. Then for any $x \in H$ there exists a nearest point $y \in K$. Hint: deduce the following inequality from the parallelogram law:

$$||y_1 - y_2||^2 \leq 2(||x - y_1||^2 + ||x - y_2||^2) - 4 \left| \left| x - \frac{y_1 + y_2}{2} \right| \right|^2.$$ 

3.3. Let $H_1 \subset H$ be a closed subspace and $x \notin H_1$. If $x_1$ is the nearest point to $x$ in $H_1$, then $x - x_1 \in H_1^\perp$.

3.4. Let $H_1 \subseteq H$ be a closed subspace and $H_2 = H_1^\perp$. Then $H = H_1 + H_2$.

3.5. Each linear continuous functional $f$ on a Hilbert space $H$ has the form $f(x) = \langle y, x \rangle$, where $y \in H$. In particular, $H^* = H$. Hint: choose the nearest point to the origin in the hyperplane $f = 1$.

3.6. **Bessel’s inequality.** Let $\{x_\alpha\}$ be an orthonormal system. Then

$$\langle \forall x \in H \rangle \sum ||\langle x, x_\alpha \rangle||^2 \leq \langle x, x \rangle.$$ 

In particular, $\langle x, x_\alpha \rangle \neq 0$ only for finite or countable number of $\alpha$. Hint: for any finite subset $A$ of indices, consider $x' = \sum_{\alpha \in A} \langle x, x_\alpha \rangle x_\alpha$. Then $x = x' + x''$ with $x' \perp x''$. Deduce that

$$\langle x, x' \rangle = \langle x', x' \rangle \leq \langle x', x' \rangle + \langle x'', x'' \rangle = \langle x, x \rangle.$$ 

3.7. **Parseval’s identity.** A system $\{x_\alpha\}$ is called complete if $\{x_\alpha\}^\perp = 0$. For any complete system and any $x \in H$, we have

$$\sum ||\langle x, x_\alpha \rangle||^2 = \langle x, x \rangle.$$ 

The series $\sum \langle x, x_\alpha \rangle x_\alpha$ converges to $x$.

3.8. A system $\{x_\alpha\}$ is said to be a Hilbert basis of $H$ if for every $x \in H$ there exists a unique convergent series

$$x = \sum c_\alpha x_\alpha.$$ 

This series is called the Fourier series for $x$. The coefficients $c_\alpha$ are called the Fourier coefficients.
3.9. Orthogonalization. Let \( \{x_n\} \) be a sequence of linearly independent elements in \( H \). There exists an orthonormal system \( \{y_n\} \) such that \( y_n \) belongs to the linear span of \( x_1, \ldots, x_n \) for every \( n \). The element \( y_n \) is unique up to a sign.

3.10. Any separable Hilbert space admits a countable complete orthonormal system. Every two infinite-dimensional separable Hilbert spaces are isomorphic.

4. Bounded and compact operators

Let \( V \) and \( W \) be normed spaces. A linear operator \( A : V \to W \) is called bounded if the image of the unit ball under \( A \) is a bounded subset of \( W \). For a bounded operator \( A \), one defined the norm

\[
||A|| = \sup_{x \in V - \{0\}} \frac{||A(x)||}{||x||}.
\]

Any bounded operator \( A \) is clearly continuous. An operator \( A \) is said to be compact if \( A(B) \) is compact, where \( B \) is the unit ball in \( V \). In other terms, \( A \) is compact iff the \( A \)-image of any bounded sequence contains a convergent subsequence.

Problems and theorems.

4.1. The set of invertible bounded operators on a Banach space is open. Hint: If \( ||A|| < 1 \), then \( (E - A)^{-1} = E + A + A^2 + \ldots \).  

4.2. Let \( C : V \to V \) be a compact operator, and \( B : V \to V \) a bounded operator. Then the operators \( CB \) and \( BC \) are compact. 

4.3. The limit of any convergent sequence of compact operators is compact. Hint: let \( T_n \) be compact operators such that \( ||T_n - T|| \to 0 \). Take any bounded sequence of vectors \( x_m \). Using the diagonal argument, choose a subsequence \( x_{m_k} \) such that \( T_n(x_{m_k}) \) converges for all \( n \). Show that \( T(x_{m_k}) \) is a Cauchy sequence.

4.4. We say that an operator \( T : V \to V \) has finite rank if the image \( T(V) \) is finite-dimensional. Any operator of finite rank is compact.

4.5. Let \( H \) be a Hilbert space. An operator \( T : H \to H \) is compact iff there is a sequence \( T_n \) of operators of finite rank such that \( ||T_n - T|| \to 0 \). Hint: Consider a Hilbert basis \( (e_n) \) of \( H \). Let \( Q_n \) be the projector to the subspace spanned by \( e_k \) with \( k > n \). If \( T \) is compact, then \( ||Q_nT|| \to 0 \).

4.6. Suppose that \( T : H \to H \) diagonalizes in some orthonormal basis, and let \( \lambda_k \) be the eigenvalues of \( T \). Show that \( T \) is compact iff \( |\lambda_k| \to 0 \).

4.7. Let \( \lambda \neq 0 \) be an eigenvalue of a compact operator \( T : H \to H \). Show that the corresponding eigenspace is finite-dimensional.

4.8. Let \( T : H \to H \) be a self-adjoint operator. Eigenvectors with different eigenvalues are orthogonal.

4.9. Let \( T : H \to H \) be a compact self-adjoint operator and \( \varepsilon > 0 \). The subspace of \( H \) spanned by all eigenvectors with eigenvalues \( \geq \varepsilon \) is finite-dimensional.

4.10. For a self-adjoint operator \( T \),

\[
||T|| = \sup\{||Tx|| : ||x|| = 1\}.
\]
4.11. If \( T : H \to H \) is a compact operator and \( T \neq 0 \), then \( ||T|| \) or \(-||T||\) is an eigenvalue of \( T \).

**Hint:** Choose a sequence \( x_n \in H \) such that \( ||x_n|| = 1 \), \( \langle Tx_n, x_n \rangle \to \pm||T|| \), and \( Tx_n \to y \). Prove that \( y \) is an eigenvector of \( T \).

4.12. **The spectral theorem.** Any compact self-adjoint operator on a Hilbert space diagonalizes in an orthonormal basis. **Hint:** let \( V \) be the subspace spanned by all eigenvectors of \( T \). Show that \( V = H \).

5. **\( L^p \) spaces**

Let \((X, \mu)\) be a measure space. Denote by \( L^p(X, \mu) \) the space of equivalence classes of measurable functions \( f \) on \( X \) s.t. \( |f|^p \) is integrable. The space \( L^p \) has the norm

\[ ||f||_p = \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}.\]

**Problems and theorems.**

5.1. Suppose that \( a \geq 0, b \geq 0 \) and \( p, q \geq 1 \) are s.t. \( 1/p + 1/q = 1 \). Then \( ab \leq a^p/b + b^q/q \). **Hint:** First prove that \( e^{\lambda x + \mu y} \leq \lambda e^x + \mu e^y \) for \( \lambda, \mu \geq 0 \) such that \( \lambda + \mu = 1 \) and all \( x, y \in \mathbb{R} \). This follows from the convexity of the exponential function. Next set \( x = p \log a, y = q \log b, \lambda = 1/p, \mu = 1/q \).

5.2. **The Hölder inequality.** \( |f_X f g d\mu| \leq ||f||_p ||g||_q \), where \( 1/p + 1/q = 1 \). **Hint:** it suffices to assume that \( ||f||_p = ||g||_q = 1 \). Integrate the inequality \( |fg| \leq ||f||_p ||g||_q \).

5.3. Deduce from the Hölder inequality that

\[ \left| \int |f|^{p-1} g \right| \leq ||f||_p^{p-1} ||g||_p.\]

5.4. **The Minkowski inequality.** The function \( ||\cdot||_p \) is a norm. **Hint:** we need to prove that

\[ ||f + \lambda g||_p \leq ||f||_p + \lambda ||g||_p \]

for all \( \lambda \geq 0 \). Compare the \( \lambda \)-derivatives of the both sides.

5.5. The space \( L^q(\mathbb{R}) \) is the dual to \( L^p(\mathbb{R}) \) for \( 1 \leq p < \infty \). **Hint:** Let \( F \in (L^p(\mathbb{R}))^* \). Define the charge \( \nu(A) = F(\chi_A) \). Prove that it is absolutely continuous. Use the Radon–Nikodym theorem.

5.6. If \( X \) is infinite, then \( L^1 \) is not \((L^\infty)^*\). **Hint:** use the Hahn-Banach theorem to extend the functional lim.

5.7. Any space \( L^p \) is Banach for \( 1 \leq p \leq \infty \).

6. **Spaces of smooth functions**

For a compact metric space \( X \), let \( C(X) \) denote the space of all continuous functions on \( X \) with the uniform norm. Let \( \Omega \subset \mathbb{R}^n \) be an open subset. Define \( C^r(\Omega) \) as the space of all functions \( f \) s.t. for \( |l| \leq r \) the derivative \( \partial^l f \) is defined in \( \Omega \) and extends to \( \Omega \) by continuity (\( l \) is a multi-index).

**Problems and theorems.**

6.1. For a compact metric space \( X \), the space \( C(X) \) is a separable Banach space.

6.2. **Dini’s lemma.** Consider an increasing sequence of functions in \( C^R(X) \). If this sequence converges point-wise, then it converges everywhere.
6.3. Let $I$ be an ideal in $C(X)$. Denote by $Z_I$ the set of all points $x \in X$ such that $f(x) = 0$ for all $f \in I$. If $Z_I = \emptyset$, then there is a function $g \in I$ such that $g > 0$ everywhere on $X$.

6.4. An ideal in $C(X)$ is maximal iff it has the form

$$I_a = \{ g \in C(X) \mid g(a) = 0 \}$$

for some $a \in X$.

6.5. Prove that the closure of any ideal $I$ in $C(X)$ coincides with

$$\{ f \in C(X) \mid f(x) = 0 \ \forall x \in Z_I \}.$$  

*Hint:* Let $f \in C(X)$ and $K$ be the set $\{ x \in X \mid |f(x)| \geq \varepsilon \}$. Prove that there exists a non-negative function $g \in I$ such that $g > 0$ on $K$. Consider the sequence $f_n = fng/(1 + ng)$.

6.6. Each Banach space $V$ is isomorphic to a closed subspace of $C(X)$. If $V$ is separable, then one can take $X = [0, 1]$. *Hint:* Let $X$ be the unit ball in $V^*$ with *-weak topology. Then $V \hookrightarrow C(X)$. If $V$ is separable, then there exists a continuous surjection $[0, 1] \twoheadrightarrow X$.

6.7. $C[0, 1]^* = V[0, 1]$ (functions of bounded variation with the norm $\|g\| = \text{Var}_0^1(g)$). *Hint:* Let $F_t(f) = \int_0^1 f dg$ in the sense of Lebesgue–Stieltjes.

6.8. Suppose that $X$ has at least two elements. Let $V \subset C^\infty(X)$ be a vector subspace such that for each pair of distinct elements $x, y \in X$ there is a function $f \in V$ such that $f(x) \neq f(y)$. Such a subspace $V$ is called separating. Prove that for any $\alpha, \beta \in \mathbb{R}$, there exists $g \in V$ such that $g(x) = \alpha$ and $g(y) = \beta$.

6.9. There exists a sequence $P_n$ of polynomials that converges uniformly on $[-1, 1]$ to the function $x \mapsto |x|$.

6.10. Let $A \subset C(X)$ be a closed subalgebra containing constants. Then $A$ is stable under the operations $(f, g) \mapsto \min(f, g)$ and $(f, g) \mapsto \max(f, g)$. *Hint:* Using the preceding problem, first show that $A$ is stable under $f \mapsto |f|$. Then use the formula

$$\min(a, b) = \frac{a + b - |a - b|}{2}.$$  

6.11. **The Stone–Weierstrass theorem.** Let $A \subset C(X)$ be a separating subalgebra containing constants. Then $A$ is dense in $C(X)$. *Hint:* let us approximate $f \in C(X)$ by elements of $A$. First fix $x \in X$. For any $y \in X$, choose $u_{x,y} \in A$ so that $u_{x,y}(x) = f(x)$ and $u_{x,y}(y) = f(y)$. Show that

$$f = \inf_x \sup_y u_{x,y}.$$  

Using compactness, replace the infimum and supremum by minimum and maximum of finitely many functions to approximate $f$ with a given precision.

6.12. A function $f : X \to \mathbb{R}$ is said to be Lipschitz if $|f(x) - f(y)| \leq C d(x, y)$ for some uniform constant $C > 0$. Lipschitz functions are dense in $C(X)$.

6.13. If $X$ is a compact subset of $\mathbb{R}^n$, then polynomials are dense in $C(X)$.

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